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LINEAR DYNAMIC EQUATIONS ON TIME SCALES

by

Kirsten R. Messer

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfillment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

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May, 2003

LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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University of Nebraska, 2003

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The theory of time scales was introduced by Stefan Hilger in his 1988 PhD dissertation, [18]. The study of dynamic equations on time scales unifies and extends the fields of differential and difference equations, highlighting the similarities and providing insight into some of the differences.

In his dissertation, [18], Hilger introduced the notion of the "delta-derivative" on a time scale. An analogous concept, the "nabla-derivative" was developed and explored by Ferhan Atici and Gusein Guseinov in [4]. It is interesting to look at what happens when these two kinds of derivatives are present in the same equation. The interaction between them yields some fascinating behavior, which in some cases is "cleaner" than the behavior found with only one type of derivative.

In Chapter 2, we examine the second-order, self-adjoint dynamic equation which contains both delta- and nabla-derivatives. We develop a reduction of order theorem, explore oscillation and disconjugacy, and look at Riccati techniques. The material in Chapter 2 has been previously published in [23] and [22].

In Chapter 3, we look at solution techniques for linear dynamic equations which can be written in a factored form. We develop complete results in the case where the equation contains only one kind of derivative. We briefly discuss the mixed derivative case deferring further consideration to later work. The material in Chapter 3 has been previously published in [21].

In the final chapter, Chapter 4, we return to the self-adjoint equation. Here,

we consider the matrix form of the equation. As in the scalar case, we develop a reduction of order theorem and explore Riccati techniques, culminating with the proof of Jacobi's Condition.

Throughout much of this dissertation, the interaction between the delta- and nabla-derivatives plays a key role. In many cases, it is rather startling to see how all of the pieces fit together. It is our hope that this work will inspire further exploration in this area.

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Contents

Chapter 1

Introduction

The theory of time scales was introduced by Stefan Hilger in his 1988 PhD dissertation, [18]. Scholars in the fields of differential equations and difference equations have long been aware of the startling similarities and intriguing differences between the two fields. The study of dynamic equations on time scales unifies and extends the fields of differential and difference equations, highlighting the similarities and providing insight into some of the differences.

In his dissertation, [18], Hilger introduced the notion of the "delta-derivative" (or Δ-derivative) on a time scale. This derivative is a generalization of both the usual derivative from differential equations and the forward difference operator from difference equations. An analogous concept, the "nabla-derivative" (or ∇-derivative) was developed and explored by Ferhan Atici and Gusein Guseinov in [4]. The nabla-derivative is a generalization of the usual derivative and the *backward* difference operator. Study of the nabla-derivative alone is not particularly revealing, since the results are usually directly analogous to results developed for the delta-derivative. What *is* interesting, however, is to look at what happens when these two kinds of derivatives are present in the same equation. The interaction between these two derivatives yields some fascinating behavior, which in some cases is "cleaner" than the behavior found when only one type of derivative is considered.

In Chapter 1 of this work, we provide a brief introduction to the calculus on a time scale, including both the delta and nabla derivatives, for the reader who may be unfamiliar with the field. We also summarize some of the key results concerning "generalized exponential functions", which play a key role in the study of dynamic equations on time scales. The material in this chapter is taken from existing sources, primarily [6, 7, 4, 3], which, in turn, rely on previously published works by Hilger and others, and we refer the interested reader to those sources for a more complete introduction.

In Chapter 2, we examine the second-order, self-adjoint dynamic equation

$$[p(t)x^{\Delta}(t)]^{\nabla} + q(t)x(t) = 0$$

on a time scale. Although the similar equation,

$$[p(t)x^{\Delta}]^{\Delta} + q(t)x^{\sigma} = 0,$$

has been studied extensively, (see, for example [9, 10, 11, 5, 12, 15, 13, 17]), little work has been done on this equation, which combines both the delta and nabla derivatives. We begin by establishing several results concerning the interaction of these two derivatives. Also included in the first section is a theorem which shows that under certain conditions, the generalized exponential functions $e_p(t, t_0)$ and $\hat{e}_r(t, t_0)$ can be related to one another. We next look at three second-order linear equations and demonstrate that they can be written in self-adjoint form. The first results which are directly related to the self-adjoint equation are contained in Section 2.3, which culminates with a reduction of order theorem. We turn our attention to oscillation and disconjugacy in Section 2.4, where we establish an analogue of the Sturm Separation Theorem, and, via the Pólya and Trench factorizations demonstrate the existence of recessive and dominant solutions of the self-adjoint equation. The final section of the chapter, Section 2.5, discusses Riccati techniques as they relate to the self-adjoint equation. The material in Chapter 2 has been previously published in [23] and [22].

We alter our focus slightly in Chapter 3. There, we look at solution techniques for linear dynamic equations on time scales which can be written in a factored form. As special cases, we obtain solution techniques for constant coefficient dynamic equations and for Euler-Cauchy dynamic equations. These special cases are consistent with results developed by Akin-Bohner and Bohner in [7, 2]. We consider equations containing either delta-derivatives or nabla-derivatives, and develop complete solution techniques in these cases. We briefly discuss second-order equations containing both delta and nabla derivatives, and defer consideration of higher order equations with a mixture of the two kinds of derivatives to later work. We conclude Chapter 3 by examining another equation which cannot itself be written in factored form, but is equivalent to one which can. The material in Chapter 3 has been previously published in [21].

In the final chapter, Chapter 4 we return to the self-adjoint equation with mixed derivatives. In this chapter, we consider the matrix form of the equation. As in the scalar case, the related scalar equation which contains only delta-derivatives has been studied extensively (see, for example [1, 14, 15, 13]).Many of the results in this chapter are analogous to the results obtained in Chapter 2 for the scalar case. We begin by establishing the Lagrange Identity for this equation, which leads to the development of Abel's Formula and a reduction of order theorem. From there, we move into a section on Riccati techniques, where we develop the relationship between the self-adjoint matrix equation, and it's related Riccati equation. Chapter 4 culminates with the proof of Jacobi's Condition, which is much more difficult to establish in the matrix case than the scalar case.

Throughout much of this dissertation, the interaction between the delta and nabla derivatives plays a key role. Although there are some theorems where the interaction

does not have much impact, in most cases, it is rather startling to see how all of the pieces fit together. It is our hope that this work will inspire further exploration in this area.

1.1 The Calculus on a Time Scale

A time scale is simply a closed subset of $\mathbb{R}$. Throughout this work, we will use the symbol $\mathbb{T}$ to represent a time scale. A given time scale is assumed to have the topology which it inherits as a subspace of $\mathbb{R}$ with the usual topology. The notation $[a, b]$ is understood to mean the real interval $[a, b]$ intersected with $\mathbb{T}$. Open and half-open intervals are understood similarly. We begin by introducing the forward and backward jump operators on $\mathbb{T}$.

Definition 1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the *forward jump operator*, $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

If $\{s \in \mathbb{T} : s > t\} = \emptyset$, (i.e. if $t = \max \mathbb{T}$), we take $\sigma(t) = t$.

Similarly, we define the *backward jump operator*, $\rho(t) : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

and, if $t = \min \mathbb{T}$, we take $\rho(t) = t$.

If $f : \mathbb{T} \to \mathbb{R}$, then the notation $f^{\sigma}(t)$ is understood to mean $f(\sigma(t))$, and $f^{\rho}(t)$ is understood to mean $f(\rho(t))$.

Points in $\mathbb{T}$ are classified as follows: If $\sigma(t) > t$, we say t is *right-scattered.* If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, we say t is *right-dense.* If $\rho(t) < t$, we say t is *left-scattered*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, we say t is *left-dense.* Points that are both right and left scattered are called *isolated*, and points that are both right and left dense are called *dense.*

Definition 2. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense-continuous* or *rd-continuous* provided f is continuous at all right-dense points in $\mathbb{T}$, and provided $\lim_{s \to t} f(s)$ exists and is finite at all left-dense $t \in \mathbb{T}$.

We say f is *left-dense-continuous* or *ld-continuous* provided f is continuous at all left-dense points in $\mathbb{T}$, and provided $\lim_{s \to t} f(s)$ exists and is finite at all right-dense $t \in \mathbb{T}$.

Definition 3. The *graininess function*, $\mu(t)$, is defined by

$$\mu(t) := \sigma(t) - t.$$

The *backward graininess function*, $\nu(t)$ is defined by

$$\nu(t) := t - \rho(t).$$

Remark 4. Recently, there has been some disagreement regarding the most appropriate definition of $\nu(t)$. In this work, we retain the original definition, which is consistent with previously published literature on ∇-derivatives. It is inconsistent, however with the current work on α-derivatives. When working with α-derivatives, the α-graininess, μ_α is defined to be $\mu_\alpha := \alpha(t) - t$. When $\alpha(t) = \rho(t)$, then, we would have $\mu_\rho := \rho(t) - t = -\nu(t)$. This inconsistency is unfortunate, but we feel it is more important that we remain consistent with the way $\nu(t)$ was defined in previously published work. To minimize confusion, we recommend the notation $\mu_\rho(t) = \rho(t) - t$ be used in work that is to be interpreted in the more general α-derivative setting.

Definition 5. The set $\mathbb{T}^\kappa$ is defined as follows. If $\mathbb{T}$ has a left-scattered maximum, M, then $\mathbb{T}^\kappa := \mathbb{T} \setminus \{M\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Similarly, if $\mathbb{T}$ has a right-scattered minimum, m, we define the set $\mathbb{T}_\kappa := \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}_\kappa = \mathbb{T}$.

Definition 6. Assume $f : \mathbb{T} \to \mathbb{R}$, and let $t \in \mathbb{T}^\kappa$. Then we define the *delta-derivative of f at t*, denoted $f^\Delta(t)$ to be the number (provided that it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood, U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$, then we say f is delta-differentiable, and we call $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ the delta-derivative of f on $\mathbb{T}^\kappa$. It is straightforward to show that for $t \in \mathbb{T}^\kappa$, f^Δ is well defined. Note, however that if $\mathbb{T}$ has a left-scattered maximum, M, then $f^\Delta(M)$ is not uniquely determined. This is precisely why we eliminate this type of point in our set $\mathbb{T}^\kappa$.

The nabla-derivative is defined in similar fashion to the delta-derivative.

Definition 7. Assume $f : \mathbb{T} \to \mathbb{R}$, and let $t \in \mathbb{T}_\kappa$. Then we define the *nabla-derivative of f at t*, denoted $f^\nabla(t)$ to be the number (provided that it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood, U of t such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|$$

for all $s \in U$.

If $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$, then we say f is nabla-differentiable, and we call $f^\nabla : \mathbb{T}_\kappa \to \mathbb{R}$ the nabla-derivative of f on $\mathbb{T}_\kappa$.

Note that in the case $\mathbb{T} = \mathbb{R}$, both the delta and nabla derivatives are simply the usual derivative. If $\mathbb{T} = \mathbb{Z}$, the delta derivative is the forward difference operator, while the nabla derivative is the backward difference operator. Both the delta and nabla derivatives possess many useful properties. We summarize some of them here.

Theorem 8. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then*

(i) If f is delta-differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is delta-differentiable at t, and

$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t}.$$

(iii) If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is delta-differentiable at t, then

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t).$$

Theorem 9. *Assume $g : \mathbb{T} \to \mathbb{R}$ is a function, and let $t \in \mathbb{T}_{\kappa}$. Then*

(i) If g is nabla-differentiable at t, then g is continuous at t.

(ii) If g is continuous at t and t is left-scattered, then g is nabla-differentiable at t, and

$$g^{\nabla}(t) = \frac{g(t) - g^{\rho}(t)}{t - \rho(t)}.$$

(iii) If t is left-dense, then g is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{g(t) - g(s)}{t - s}$$

exists as a finite number. In this case

$$g^{\nabla}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) If g is nabla-differentiable at t, then

$$g^{\rho}(t) = g(t) - \nu(t) g^{\nabla}(t).$$

Looking at properties *(ii)* and *(iii)* in each of the theorems above gives us a more intuitive understanding of these derivatives than can be gained via the definition alone. In the case of the delta-derivative, if $t \in \mathbb{T}$ is right-dense, then the delta-derivative behaves in much the same way as the usual derivative. It can be regarded

as the slope of the tangent line to the function at t, although if t is both right-dense and left-scattered, the limit is a one-sided limit. On the other hand, if t is right-scattered, then the $f^{\Delta}(t)$ is the slope of the line segment connecting $f(t)$ and $f(\sigma(t))$. In this case, it makes no difference what the behavior of the function is like to the left of t, beyond the requirement that f be continuous at t. So, the delta derivative provides a mixture of the discrete behavior of the forward difference operator, and the continuous behavior of the usual derivative. The nabla derivative can be interpreted similarly, but in that case the focus is on what happens to the left of a given point, rather than the right.

It is easy to show that both the delta and nabla derivatives are linear. There are also product and quotient rules for both derivatives.

Theorem 10. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are delta-differentiable at $t \in \mathbb{T}^{\kappa}$. Then*

(i) The product fg is delta-differentiable at t, with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) = f^{\Delta}(t)g^{\sigma}(t) + f(t)g^{\Delta}(t).$$

(ii) If $g(t)g^{\sigma}(t) \neq 0$, then $\frac{f}{g}$ is delta-differentiable at t, with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}.$$

Theorem 11. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla-differentiable at $t \in \mathbb{T}_{\kappa}$. Then*

(i) The product fg is nabla-differentiable at t, with

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f^{\rho}(t)g^{\nabla}(t) = f^{\nabla}(t)g^{\rho}(t) + f(t)g^{\nabla}(t).$$

(ii) If $g(t)g^{\rho}(t) \neq 0$, then $\frac{f}{g}$ is nabla-differentiable at t, with

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t)g^{\nabla}(t)}{g(t)g^{\rho}(t)}.$$

Of course, the calculus on a time scale would not be complete without a concept of integration to complement the derivative. It can be shown that if a function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, then it has a delta antiderivative. Similarly, if $g : \mathbb{T} \to \mathbb{R}$ is ld-continuous, then it has a nabla antiderivative. We then define the delta and nabla integrals of f and g in terms of these antiderivatives.

Definition 12. Assume $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, and let F be a delta antiderivative of f. That is, suppose $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. Then the *indefinite delta integral* of f is given by

$$\int f(t)\, \Delta t = F(t) + C.$$

The *delta Cauchy integral* is defined by

$$\int_r^s f(t)\ \Delta t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}.$$

Definition 13. Assume $g : \mathbb{T} \to \mathbb{R}$ is ld-continuous, and let G be a nabla antiderivative of g. That is, suppose $G^{\nabla}(t) = g(t)$ for all $t \in \mathbb{T}_{\kappa}$. Then the *indefinite nabla integral* of g is given by

$$\int g(t)\ \nabla t = G(t) + C.$$

The *nabla Cauchy integral* is defined by

$$\int_r^s g(t)\ \nabla t = G(s) - G(r) \quad \text{for all} \quad r, s \in \mathbb{T}.$$

These definitions can be generalized to apply to a more broad category of functions called *regulated* functions. See, for example, [6, Section 1.4]. Since our applications of integration usually fall within the more restrictive definition, however, we will not dwell on the somewhat technical details of this more general definition.

As with the derivatives, both integrals are linear. Additional properties are given in the following theorems.

Theorem 14. *If a, b, $c \in \mathbb{T}$, and f, g are rd-continuous, then*

(i) $\int_a^b f(t)\ \Delta t = -\int_b^a f(t)\ \Delta t.$

(ii) $\int_a^b f(t)\ \Delta t = \int_a^c f(t)\ \Delta t + \int_c^b f(t)\ \Delta t.$

(iii) $\int_a^b f^{\sigma}(t) g^{\Delta}(t)\ \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(t)\ \Delta t.$

(iv) $\int_a^b f(t) g^{\Delta}(t)\ \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g^{\sigma}(t)\ \Delta t.$

Theorem 15. *If a, b, $c \in \mathbb{T}$, and f, g are ld-continuous, then*

(i) $\int_a^b f(t)\ \nabla t = -\int_b^a f(t)\ \nabla t.$

(ii) $\int_a^b f(t)\ \nabla t = \int_a^c f(t)\ \nabla t + \int_c^b f(t)\ \nabla t.$

(iii) $\int_a^b f^{\rho}(t) g^{\nabla}(t)\ \nabla t = (fg)(b) - (fg)(a) - \int_a^b f^{\nabla}(t) g(t)\ \nabla t.$

(iv) $\int_a^b f(t) g^{\nabla}(t)\ \nabla t = (fg)(b) - (fg)(a) - \int_a^b f^{\nabla}(t) g^{\rho}(t)\ \nabla t.$

1.2 Generalized Exponential Functions

In the study of differential equations, the properties of the exponential function e^{at} are crucial to many of the standard results. In difference equations, there is a similar reliance on functions of the form a^t. In this section, we look at *generalized exponential functions* on time scales, which will play this key role in the time scales context.

Definition 16. We say that a function $r : \mathbb{T} \to \mathbb{R}$ is *regressive* if

$$1 + \mu(t)r(t) \neq 0$$

for all $t \in \mathbb{T}^\kappa$.
We further define the sets $\mathcal{R}$ and $\mathcal{R}^+$ by

$$\mathcal{R} := \{r : \mathbb{T} \to \mathbb{R} \mid r \text{ is rd-continuous and regressive}\}$$

and

$$\mathcal{R}^+ := \{r \in \mathcal{R} \mid 1 + \mu(t)r(t) > 0 \text{ for all } t \in \mathbb{T}\}$$

We now define the "circle plus" addition, $\oplus$, on $\mathcal{R}$. If p, $q \in \mathcal{R}$, then

$$p \oplus q := p + q + \mu pq,$$

where μ is the graininess function. It can be shown that $\mathcal{R}$ is an Abelian group under this circle plus addition. For this reason, we refer to $\mathcal{R}$ as the regressive group. The family $\mathcal{R}^+$ of positively regressive functions is a subgroup of $\mathcal{R}$ under circle plus addition.

If $p \in \mathcal{R}$, then the additive inverse of p under circle plus addition is denoted $\ominus p$, and is given by

$$\ominus p = \frac{-p}{1 + \mu p}.$$

For p, $q \in \mathcal{R}$, we write $p \ominus q$ to denote $p \oplus (\ominus q)$.

If $r \in \mathcal{R}$, then the initial value problem

$$y^\Delta = r(t)y, \quad y(t_0) = 1$$

has a unique solution. We denote this solution by

$$e_r(\cdot, t_0),$$

and call it the generalized (delta) exponential function. It can be shown that if $r \in \mathcal{R}^+$, then $e_r(\cdot, t_0) > 0$ for all $t \in \mathbb{T}$.

Some of the key properties of the generalized (delta) exponential function are given here.

Theorem 17. *If p, $q \in \mathcal{R}$, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$*;*

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$*;*

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$*;*

(viii) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$

A generalized nabla exponential function can be developed in an analogous fashion. We state the relevant definitions and properties here.

Definition 18. We say that a function $r : \mathbb{T} \to \mathbb{R}$ is *ν-regressive* if

$$1 - \nu(t)r(t) \neq 0$$

for all $t \in \mathbb{T}_\kappa$.
We further define the sets $\mathcal{R}_\nu$ and $\mathcal{R}_\nu^+$ by

$$\mathcal{R}_\nu := \{r : \mathbb{T} \to \mathbb{R} \mid r \text{ is ld-continuous and } \nu-\text{regressive}\}$$

and

$$\mathcal{R}_\nu^+ := \{r \in \mathcal{R}_\nu \mid 1 - \nu(t)r(t) > 0 \text{ for all } t \in \mathbb{T}\}$$

The set $\mathcal{R}_\nu$ is an Abelian group under the operation $\oplus_\nu$, which is defined by

$$p \oplus_\nu q := p + q - \nu pq,$$

for p, $q \in \mathcal{R}_\nu$. The family $\mathcal{R}_\nu^+$ is a subgroup.
If $p \in \mathcal{R}_\nu$, then the additive inverse of p under $\oplus_\nu$ is denoted $\ominus_\nu p$, and is given by

$$\ominus_\nu p = \frac{-p}{1-\nu p}.$$

For p, $q \in \mathcal{R}_\nu$, we write $p \ominus_\nu q$ to denote $p \oplus_\nu (\ominus_\nu q)$.
If $r \in \mathcal{R}_\nu$, then the initial value problem

$$y^{\nabla} = r(t)y, \quad y(t_0) = 1$$

has a unique solution. We denote this solution by

$$\hat{e}_r(\cdot, t_0),$$

and call it the generalized nabla exponential function. It can be shown that if $r \in \mathcal{R}_\nu^+$, then $\hat{e}_r(\cdot, t_0) > 0$ for all $t \in \mathbb{T}$.

Some of the key properties of the generalized nabla exponential function are given here.

Theorem 19. *If p, $q \in \mathcal{R}_\nu$, then*

(i) $\hat{e}_0(t,s) \equiv 1$ *and* $\hat{e}_p(t,t) \equiv 1$;

(ii) $\hat{e}_p(\rho(t), s) = (1 - \nu(t)p(t))\hat{e}_p(t,s)$;

(iii) $\frac{1}{\hat{e}_p(t,s)} = \hat{e}_{\ominus_\nu p}(t,s)$;

(iv) $\hat{e}_p(t,s) = \frac{1}{\hat{e}_p(s,t)} = \hat{e}_{\ominus_\nu p}(s,t)$;

(v) $\hat{e}_p(t,s)\hat{e}_p(s,r) = \hat{e}_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus_\nu q}(t,s)$;

(vii) $\frac{\hat{e}_p(t,s)}{\hat{e}_q(t,s)} = \hat{e}_{p\ominus_\nu q}(t,s)$;

(viii) $\left(\frac{1}{\hat{e}_p(\cdot,s)}\right)^\nabla = -\frac{p(t)}{e_p^\rho(\cdot,s)}$

Chapter 2

A Second-Order Self-Adjoint Dynamic Equation on a Time Scale

2.1 Preliminary Results

This chapter is concerned with the study of the second-order self-adjoint equation $[p(t)x^{\Delta}]^{\nabla}+q(t)x=0$ on a time scale. Since the equation we are interested in contains both Δ and ∇-derivatives, we will want to know how these two different derivatives interact. Toward this end, we will need an analog of L'Hôpital's Rule. A version of this crucial theorem for Δ-derivatives, is contained in [6], although it is presented here in a slightly different form (Theorem 22). We then develop L'Hôpital's rule for ∇-derivatives (Theorem 23).

We may want to employ L'Hôpital's Rule to evaluate a limit as $t \to \pm\infty$, so we make the following definitions.

Definition 20. Let $\varepsilon > 0$. If $\mathbb{T}$ is unbounded above, we define a *left neighborhood of* ∞, which we denote by $L_\varepsilon(\infty)$, by

$$L_\varepsilon(\infty) = \left\{t \in \mathbb{T} : t > \frac{1}{\varepsilon}\right\}.$$

Similarly, if $\mathbb{T}$ is unbounded below, we define a *right neighborhood of* $-\infty$, denoted $R_\varepsilon(-\infty)$ by

$$R_\varepsilon(-\infty) = \left\{t \in \mathbb{T} : t < -\frac{1}{\varepsilon}\right\}.$$

We next define right and left neighborhoods for points in $\mathbb{T}$.

Definition 21. Let $\varepsilon > 0$. For any right-dense $t_0 \in \mathbb{T}$, define a *right neighborhood of* t_0, denoted $R_\varepsilon(t_0)$, by

$$R_\varepsilon(t_0) := \{t \in \mathbb{T} : 0 < t - t_0 < \varepsilon\}.$$

Similarly, for any left-dense $t_0 \in \mathbb{T}$, define a *left neighborhood of* t_0, denoted $L_\varepsilon(t_0)$, by

$$L_\varepsilon(t_0) := \{t \in \mathbb{T} : 0 < t_0 - t < \varepsilon\}.$$

Theorem 22 (L'Hôpital's Rule for Δ-derivatives). *Assume f and g are Δ-differentiable on $\mathbb{T}$, and let $t_0 \in \mathbb{T} \cup \{\infty\}$. If $t_0 \in \mathbb{T}$, assume t_0 is left-dense. Furthermore, assume*

$$\lim_{t \to t_0^-} f(t) = \lim_{t \to t_0^-} g(t) = 0,$$

and suppose there exists $\varepsilon > 0$ with

$$g(t)g^\Delta(t) < 0 \quad \textit{for all} \quad t \in L_\varepsilon(t_0).$$

Then we have

$$\liminf_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)} \le \liminf_{t \to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)}.$$

Theorem 23 (L'Hôpital's Rule for ∇-derivatives). *Assume f and g are ∇-differentiable on $\mathbb{T}$ and let $t_0 \in \mathbb{T} \cup \{-\infty\}$. If $t_0 \in \mathbb{T}$, assume t_0 is right-dense. Furthermore, assume*

$$\lim_{t \to t_0^+} f(t) = \lim_{t \to t_0^+} g(t) = 0,$$

and suppose there exists $\varepsilon > 0$ with

$$g(t)g^\nabla(t) > 0 \quad \textit{for all} \quad t \in R_\varepsilon(t_0).$$

Then

$$\liminf_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)} \le \liminf_{t \to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)}.$$

Proof. Without loss of generality, assume $g(t)$ and $g^\nabla(t)$ are both strictly positive on $R_\varepsilon(t_0)$.

Let $\delta \in (0, \varepsilon]$, and let $a := \inf_{\tau \in R_\delta(t_0)} \frac{f^\nabla(\tau)}{g^\nabla(\tau)}$, $b := \sup_{\tau \in R_\delta(t_0)} \frac{f^\nabla(\tau)}{g^\nabla(\tau)}$. To complete the proof, it suffices to show

$$a \le \inf_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)} \le \sup_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)} \le b,$$

as we may then let $\delta \to 0$ to obtain the desired result.

We must be careful here, as either a or b could possibly be infinite. Note, however, that since $g^\nabla(\tau) > 0$ on $R_\delta(t_0)$, we have $a < \infty$. Similarly, $b > -\infty$. So our only concern is if $a = -\infty$ or $b = \infty$. But, if $a = -\infty$, we have immediately that

$$a \le \inf_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)},$$

as desired, and if $b = \infty$, we have immediately that

$$\sup_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)} \leq b,$$

as desired. Therefore, we may assume that both a and b are finite. Then

$$ag^\nabla(\tau) \leq f^\nabla(\tau) \leq bg^\nabla(\tau) \quad \text{for all} \quad \tau \in R_\delta(t_0),$$

and by a theorem of Guseinov and Kaymakçalan [16],

$$\int_t^s ag^\nabla(\tau)\ \nabla\tau \leq \int_t^s f^\nabla(\tau)\ \nabla\tau \leq \int_t^s bg^\nabla(\tau)\ \nabla\tau \quad \text{for all} \quad s, t \in R_\delta(t_0),\ t < s.$$

Integrating, we see that

$$ag(s) - ag(t) \leq f(s) - f(t) \leq bg(s) - bg(t) \quad \text{for all} \quad s, t \in R_\delta(t_0),\ t < s.$$

Letting $t \to t_0^+$, we get

$$ag(s) \leq f(s) \leq bg(s) \quad \text{for all} \quad s \in R_\delta(t_0),$$

and thus

$$a \leq \inf_{s \in R_\delta(t_0)} \frac{f(s)}{g(s)} \leq \sup_{s \in R_\delta(t_0)} \frac{f(s)}{g(s)} \leq b.$$

Then, by the discussion above, the proof is complete. □

Remark 24. Although these theorems are only stated in terms of one-sided limits, analogous results can be established if the limit is taken from the other direction. To apply L'Hôpital's rule using Δ-derivatives and a right-sided limit, t_0 must be right-dense (or $-\infty$ if $\mathbb{T}$ is unbounded below), and gg^Δ must be strictly positive on a right neighborhood of t_0. Similarly, to apply L'Hôpital's rule using ∇-derivatives and a left-sided limit, t_0 must be left-dense (or ∞ if $\mathbb{T}$ is unbounded above), and gg^∇ must be strictly negative on some left neighborhood of t_0.

In order to determine when the two types of derivatives may be interchanged, we need to consider some of the points in our time scale separately, so let

$$A := \{t \in \mathbb{T} \mid t \text{ is left-dense and right-scattered}\}, \quad \text{and} \quad \mathbb{T}_A := \mathbb{T} \setminus A.$$

Similarly, let

$$B := \{t \in \mathbb{T} \mid t \text{ is right-dense and left-scattered}\}, \quad \text{and} \quad \mathbb{T}_B := \mathbb{T} \setminus B.$$

Lemma 25. *If $t \in \mathbb{T}_A$, then $\sigma(\rho(t)) = t$. If $t \in \mathbb{T}_B$, then $\rho(\sigma(t)) = t$.*

Proof. We will only prove the first statement. The proof of the second statement is similar. If $t \in \mathbb{T}_A$, then either t is left-scattered, or t is both left-dense and right-dense. If t is left-scattered, then $\rho(t)$ is right-scattered and it is clear that $\sigma(\rho(t)) = t$. If t is both left-dense and right-dense, then $\sigma(t) = t$ and $\rho(t) = t$. Hence $\sigma(\rho(t)) = \sigma(t) = t$. In either case we get the desired result. $\square$

Theorem 26. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$ and f^Δ is rd-continuous on $\mathbb{T}^\kappa$, then f is ∇-differentiable on $\mathbb{T}_\kappa$, and*

$$f^\nabla(t) = \begin{cases} f^\Delta(\rho(t)) & t \in \mathbb{T}_A \\ \lim_{s \to t^-} f^\Delta(s) & t \in A. \end{cases}$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}_\kappa$ and g^∇ is ld-continuous on $\mathbb{T}_\kappa$, then g is Δ-differentiable on $\mathbb{T}^\kappa$, and

$$g^\Delta(t) = \begin{cases} g^\nabla(\sigma(t)) & t \in \mathbb{T}_B \\ \lim_{s \to t^+} g^\nabla(s) & t \in B. \end{cases}$$

Proof. We will only prove the first statement. The proof of the second statement is similar. First, assume $t \in \mathbb{T}_A$. Then there are two cases: Either

(i) t is left-scattered, or

(ii) t is both left-dense and right-dense.

Case (i): Suppose t is left-scattered and f is Δ-differentiable on $\mathbb{T}^\kappa$. Then $\rho(t)$ is right-scattered, and

$$f^\Delta(\rho(t)) = \frac{f(\sigma(\rho(t))) - f(\rho(t))}{\sigma(\rho(t)) - \rho(t)}$$

Now, as f is Δ-differentiable on $\mathbb{T}^\kappa$, f is continuous on $\mathbb{T}$. Then, since t is left-scattered, f is ∇-differentiable at t, and we see that

$$\begin{aligned} f^\Delta(\rho(t)) &= \frac{f(\sigma(\rho(t))) - f(\rho(t))}{\sigma(\rho(t)) - \rho(t)} \\ &= \frac{f(t) - f(\rho(t))}{t - \rho(t)} \\ &= f^\nabla(t). \end{aligned}$$

Case (ii): Now, suppose t is both left-dense and right-dense, and $f : \mathbb{T} \to \mathbb{R}$ is continuous on $\mathbb{T}$ and Δ-differentiable at t. Since t is right-dense and f is Δ-differentiable at t, we have that

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists. But t is left-dense as well, so this expression also defines $f^\nabla(t)$, and we see

that

$$\begin{aligned} f^{\nabla}(t) &= \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \\ &= f^{\Delta}(t) \\ &= f^{\Delta}(\rho(t)). \end{aligned}$$

So, we have established the desired result in the case where $t \in \mathbb{T}_A$.

Now suppose $t \in A$. Then t is left-dense. Hence $f^{\nabla}(t)$ exists provided

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists.

As t is right-scattered, we need only consider the limit as $s \to t$ from the left. Then we apply L'Hôpital's rule [6], differentiating with respect to s to get

$$\lim_{s \to t-} \frac{f(t) - f(s)}{t - s} = \lim_{s \to t-} \frac{-f^{\Delta}(s)}{-1} = \lim_{s \to t-} f^{\Delta}(s).$$

Since we have assumed that f^{Δ} is rd-continuous, this limit exists. Hence f is ∇-differentiable, and $f^{\nabla}(t) = \lim_{s \to t-} f^{\Delta}(t)$, as desired. □

Corollary 27. *If $t_0 \in \mathbb{T}$, and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous on $\mathbb{T}$, then $\int_{t_0}^{t} f(\tau)\Delta\tau$ is ∇-differentiable on $\mathbb{T}$ and*

$$\left[\int_{t_0}^{t} f(\tau)\Delta\tau\right]^{\nabla} = \begin{cases} f(\rho(t)) & t \in \mathbb{T}_A \\ \lim_{s \to t-} f(s) & t \in A. \end{cases}$$

If $t_0 \in \mathbb{T}$, and $g : \mathbb{T} \to \mathbb{R}$ is ld-continuous on $\mathbb{T}$, then $\int_{t_0}^{t} g(\tau)\nabla\tau$ is Δ-differentiable on $\mathbb{T}$ and

$$\left[\int_{t_0}^{t} g(\tau)\nabla\tau\right]^{\Delta} = \begin{cases} g(\sigma(t)) & t \in \mathbb{T}_B \\ \lim_{s \to t+} g(s) & t \in B. \end{cases}$$

The following corollary was previously established by Atici and Guseinov in their work [4].

Corollary 28. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^{\kappa}$ and if f^{Δ} is continuous on $\mathbb{T}^{\kappa}$, then f is ∇-differentiable on $\mathbb{T}_{\kappa}$ and*

$$f^{\nabla}(t) = f^{\Delta\rho}(t) \quad \textit{for } t \in \mathbb{T}_{\kappa}.$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}^{\kappa}$ and if g^{∇} is continuous on $\mathbb{T}_{\kappa}$, then g is Δ-differentiable on $\mathbb{T}^{\kappa}$ and

$$g^{\Delta}(t) = g^{\nabla\sigma}(t) \quad \textit{for } t \in \mathbb{T}^{\kappa}.$$

Now, there are a couple more integral formulas that will be useful, the first two of which were established in [6] and [4].

Lemma 29. *The following hold:*

(i) $\int_t^{\sigma(t)} f(s)\ \Delta s = \mu(t)f(t)$

(ii) $\int_{\rho(t)}^t f(s)\ \nabla s = \nu(t)f(t)$

(iii) $\int_t^{\sigma(t)} f(s)\ \nabla s = \mu(t)f^\sigma(t)$

(iv) $\int_{\rho(t)}^t f(s)\ \Delta s = \nu(t)f^\rho(t)$

We conclude this section by exploring the relationship between the generalized exponential functions associated with the Δ and ∇ derivatives.

Lemma 30. *Let $p : \mathbb{T} \to \mathbb{R}$. Then p is regressive if and only if $-p^\rho$ is ν-regressive, and $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$ if and only if $1 + \nu(t)p^\rho(t) > 0$ for all $t \in \mathbb{T}$. Similarly, if $q : \mathbb{T} \to \mathbb{R}$, then q is ν-regressive if and only if $-q^\sigma$ is regressive, and $1 - \nu(t)q(t) > 0$ for all $t \in \mathbb{T}$ if and only if $1 - \mu(t)q^\sigma(t) > 0$ for all $t \in \mathbb{T}$.*

Proof. We will only prove the first statement. The proof of the second statement is similar.

First, assume p is regressive. We then wish to show that $1 + \nu(t)(p^\rho(t)) \neq 0$.

Case 1: Fix $t \in \mathbb{T}_A$. Then $\rho(t) \in \mathbb{T}$, and as p is regressive, we have that

$$1 + \mu(\rho(t))p(\rho(t)) \neq 0,$$

so, using the definition of $\mu(t)$,

$$1 + [\sigma(\rho(t)) - \rho(t)]p(\rho(t)) \neq 0.$$

But $t \in \mathbb{T}_A$, so $\sigma(\rho(t)) = t$, and we get

$$1 + [t - \rho(t)]p^\rho(t) \neq 0,$$

or

$$1 + \nu(t)p^\rho(t) \neq 0$$

as desired.

Case 2: Fix $t \in A$. Then t is left-dense and right-scattered, so $\nu(t) = 0$. Hence

$$1 + \nu(t)p^\rho(t) = 1 + 0p^\rho(t) = 1 \neq 0.$$

As $1 + \nu(t)p^\rho(t) \neq 0$ for any $t \in \mathbb{T}$, we see that $-p^\rho$ is ν-regressive.

Conversely, suppose $-p^\rho$ is ν-regressive. We then wish to show that $1+p(t)\mu(t) \neq 0$.

Case 1: Fix $t \in \mathbb{T}_B$. Then $\sigma(t) \in \mathbb{T}$, and, as $-p^\rho$ is ν-regressive, we have that

$$1 + \nu(\sigma(t))p^\rho(\sigma(t)) \neq 0,$$

so, using the definition of $\nu(t)$,

$$1 + [\sigma(t) - \rho(\sigma(t))]p(\rho(\sigma(t))) \neq 0.$$

But $t \in \mathbb{T}_B$, so $\rho(\sigma(t)) = t$, and we get

$$1 + [\sigma(t) - t]p(t) \neq 0$$

or

$$1 + \mu(t)p(t) \neq 0$$

as desired.

Case 2: Fix $t \in B$. Then t is right-dense and left-scattered, so $\mu(t) = 0$. Hence

$$1 + \mu(t)p(t) = 1 + 0p(t) = 1 \neq 0.$$

As $1 + \mu(t)p(t) \neq 0$ for any $t \in \mathbb{T}$, we see that p is regressive.

To show $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$ if and only if $1 + \nu(t)p^\rho(t) > 0$ for all $t \in \mathbb{T}$, simply replace "$\neq 0$" by "> 0" in the preceding proof. $\square$

Theorem 31 (Equivalence of delta and nabla exponential functions). *If p is continuous and regressive, then*

$$e_p(t, t_0) = \hat{e}_{\frac{p^\rho}{1+p^\rho\nu}}(t, t_0) = \hat{e}_{\ominus_\nu(-p^\rho)}(t, t_0).$$

If q is continuous and ν-regressive, then

$$\hat{e}_q(t, t_0) = e_{\frac{q^\sigma}{1-q^\sigma\mu}}(t, t_0) = e_{\ominus(-q^\sigma)}(t, t_0).$$

Proof. We will only prove the first statement. The proof of the second statement is similar. Suppose that $p : \mathbb{T} \to \mathbb{R}$ is continuous and regressive, then by Lemma 30 we have that $-p^\rho$ is ν-regressive. Furthermore, since p is continuous, $-p^\rho$ is ld-continuous. Hence $-p^\rho \in \mathcal{R}_\nu$. Then as $\mathcal{R}_\nu$ is an Abelian group under $\oplus_\nu$, we see that $\ominus_\nu(-p^\rho) = \frac{p^\rho}{1+p^\rho\nu} \in \mathcal{R}_\nu$, and therefore $\hat{e}_{\frac{p^\rho}{1+p^\rho\nu}}(t, t_0)$ exists.

To complete the proof, then, it suffices to show that $e_p(t, t_0)$ solves the initial value problem

$$y^\nabla = \frac{p^\rho(t)}{1 + p^\rho(t)\nu(t)}y, \quad y(t_0) = 1.$$

Let $y(t) = e_p(t, t_0)$. Then
$$y(t_0) = e_p(t_0, t_0) = 1.$$
Now, $e_p^{\Delta}(t, t_0) = p(t)e_p(t, t_0)$, which is continuous. Hence by Corollary 28, $e_p^{\nabla}(t, t_0) = e_p^{\Delta\rho}(t, t_0)$, and we get
$$\begin{aligned} y^{\nabla}(t) &= e_p^{\nabla}(t, t_0) \\ &= e_p^{\Delta\rho}(t, t_0) \\ &= p^{\rho}(t)e_p^{\rho}(t, t_0) \\ &= p^{\rho}(t)\left[e_p(t, t_0) - \nu(t)e_p^{\nabla}(t, t_0)\right]. \end{aligned}$$
Rearranging this equation gives
$$e_p^{\nabla}(t, t_0)\left[1 + \nu(t)p^{\rho}(t)\right] = p^{\rho}(t)e_p(t, t_0),$$
so
$$e_p^{\nabla}(t, t_0) = \frac{p^{\rho}(t)}{1 + \nu(t)p^{\rho}(t)}e_p(t, t_0).$$
Putting this back in terms of y, we get
$$y^{\nabla}(t) = \ominus_{\nu}(-p^{\rho})y(t),$$
and the proof is complete. $\square$

2.2 Second-order Linear Dynamic Equations

Recall that we are interested in the second-order self-adjoint dynamic equation
$$Lx = 0 \text{ where } Lx(t) = [p(t)x^{\Delta}(t)]^{\nabla} + q(t)x(t). \tag{2.1}$$
Here we assume that $p : \mathbb{T} \to \mathbb{R}$ is continuous, $q : \mathbb{T} \to \mathbb{R}$ is ld-continuous and that
$$p(t) > 0 \text{ for all } t \in \mathbb{T}.$$
Define the set $\mathbb{D}$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is continuous and such that $[p(t)x^{\Delta}]^{\nabla} : \mathbb{T}^{\kappa}_{\kappa} \to \mathbb{R}$ is ld-continuous. A function $x \in \mathbb{D}$ is said to be a solution of $Lx = 0$ on $\mathbb{T}$ provided $Lx(t) = 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Now, consider the second-order linear dynamic equations
$$M_1x = 0 \text{ where } M_1x = x^{\Delta\nabla} + p_1(t)x^{\nabla} + p_2(t)x, \tag{2.2}$$
$$M_2x = 0 \text{ where } M_2x = x^{\Delta\nabla} + a_1(t)x^{\Delta} + a_2(t)x, \tag{2.3}$$

and

$$M_3x = 0 \text{ where } M_3x = x^{\Delta\nabla} + r_1(t)x^{\nabla} + r_2(t)x^{\rho}, \tag{2.4}$$

where $p_i, a_i, r_i : \mathbb{T} \to \mathbb{R}$ are ld-continuous for $i \in \{1,2\}$. Take $\mathbb{D}_M$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that x is Δ -differentiable on $\mathbb{T}^{\kappa}$, $x^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is ∇ -differentiable on $\mathbb{T}^{\kappa}_{\kappa}$, and $x^{\Delta\nabla} : \mathbb{T}^{\kappa}_{\kappa} \to \mathbb{R}$ is ld-continuous. For $i = 1, 2, 3$, we say x is a solution of $M_ix = 0$ on $\mathbb{T}$ provided x is in $\mathbb{D}_M$, and $M_ix = 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Theorem 32. *If p_2 is ld-continuous and $p_1 \in \mathcal{R}^+_{\nu}$, then the dynamic equation* (2.2) *can be written in self-adjoint form, with*

$$p(t) = \hat{e}_{p_1}(t, t_0) \text{ and } q(t) = \hat{e}_{p_1}(t, t_0)p_2(t).$$

Furthermore, in this case, if x is a solution of (2.2) *on $\mathbb{T}$, then x is also a solution of the self-adjoint form of the equation.*

Proof. Suppose we have

$$x^{\Delta\nabla} + p_1(t)x^{\nabla} + p_2(t)x = 0.$$

Assume p_2 is ld-continuous and $p_1 \in \mathcal{R}^+_{\nu}$. Then $\hat{e}_{p_1}(t, t_0)$ is well defined and positive. Multiplying through by $\hat{e}_{p_1}(t, t_0)$, we get

$$\hat{e}_{p_1}(t, t_0)x^{\Delta\nabla} + \hat{e}_{p_1}(t, t_0)p_1(t)x^{\nabla} + \hat{e}_{p_1}(t, t_0)p_2(t)x = 0.$$

Then, since $\hat{e}_{p_1}(t, t_0)$ solves the IVP

$$y^{\nabla} = p_1(t)y, \quad y(t_0) = 1,$$

we have that

$$[\hat{e}_{p_1}(t, t_0)]^{\nabla} = p_1(t)\hat{e}_{p_1}(t, t_0).$$

So our equation becomes

$$\hat{e}_{p_1}(t, t_0)x^{\Delta\nabla} + [\hat{e}_{p_1}(t, t_0)]^{\nabla}x^{\nabla} + \hat{e}_{p_1}(t, t_0)p_2(t)x = 0.$$

Furthermore, x^{Δ} is ∇-differentiable, hence continuous, so by Corollary 28, $x^{\nabla} = x^{\Delta\rho}$ and we get

$$\hat{e}_{p_1}(t, t_0)x^{\Delta\nabla} + [\hat{e}_{p_1}(t, t_0)]^{\nabla}x^{\Delta\rho} + \hat{e}_{p_1}(t, t_0)p_2(t)x = 0.$$

Then by the product rule, we see that

$$[\hat{e}_{p_1}(t, t_0)x^{\Delta}]^{\nabla} + \hat{e}_{p_1}(t, t_0)p_2(t)x = 0.$$

This equation is in self-adjoint form with $p(t)$ and $q(t)$ as desired.

Now suppose x is a solution of (2.2), p_2 is ld-continuous and $p_1 \in \mathcal{R}^+_{\nu}$. Based on

the above development, it is clear that x satisfies the dynamic equation

$$[\hat{e}_{p_1}(t,t_0)x^{\Delta}]^{\nabla} + \hat{e}_{p_1}(t,t_0)p_2(t)x = 0.$$

Hence to show x is a solution of this dynamic equation, we need only show that $x \in \mathbb{D}$. Note that x^{Δ} is ∇ -differentiable, and therefore continuous. Also,

$$\begin{aligned} [\hat{e}_{p_1}(t,t_0)x^{\Delta}]^{\nabla} &= \hat{e}^{\nabla}_{p_1}(t,t_0)x^{\rho} + \hat{e}_{p_1}(t,t_0)x^{\Delta\nabla} \\ &= p_1(t)\hat{e}_{p_1}(t,t_0)x^{\rho} + \hat{e}_{p_1}(t,t_0)x^{\Delta\nabla}, \end{aligned}$$

which is ld-continuous, and therefore, $x \in \mathbb{D}$. □

Corollary 33. *If a_2 is ld-continuous and $-a_1 \in \mathcal{R}^+_{\nu}$, then the dynamic equation* (2.3) *can be written in self-adjoint form, with*

$$p(t) = \hat{e}_{\frac{a_1}{1+a_1\nu}}(t,t_0) \quad \text{and} \quad q(t) = \frac{a_2(t)}{1+a_1(t)\nu(t)}\hat{e}_{\frac{a_1}{1+a_1\nu}}(t,t_0).$$

Furthermore, if x is a solution of (2.3)*, then x is also a solution of the self-adjoint form of the equation.*

Proof. Suppose we have

$$x^{\Delta\nabla} + a_1(t)x^{\Delta} + a_2(t)x = 0.$$

Recall that if $f : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable, then $f(t) = f^{\rho}(t) + \nu(t)f^{\nabla}(t)$. Thus $x^{\Delta} = x^{\Delta\rho} + \nu(t)x^{\Delta\nabla}$. Making this substitution, we have

$$x^{\Delta\nabla} + a_1(t)(x^{\Delta\rho} + \nu(t)x^{\Delta\nabla}) + a_2(t)x = 0,$$

and hence

$$(1 + a_1(t)\nu(t))x^{\Delta\nabla} + a_1(t)x^{\Delta\rho} + a_2(t)x = 0.$$

Now $-a_1(t) \in \mathcal{R}^+_{\nu}$, so the leading coefficient is positive and we may divide through by it. Furthermore, as before, we have that x^{Δ} is continuous, so $x^{\nabla} = x^{\Delta\rho}$. Thus, we get

$$x^{\Delta\nabla} + \frac{a_1(t)}{(1+a_1(t)\nu(t))}x^{\nabla} + \frac{a_2(t)}{(1+a_1(t)\nu(t))}x = 0.$$

This is in the form (2.2). As a_1 and a_2 are ld-continuous, so are $\frac{a_1}{(1+a_1\nu)}$ and $\frac{a_2}{(1+a_1\nu)}$. Further,

$$\left(1 - \nu(t)\frac{a_1(t)}{1+a_1(t)\nu(t)}\right) = \frac{1+a_1(t)\nu(t) - a_1(t)\nu(t)}{1+a_1(t)\nu(t)} = \frac{1}{1+a_1(t)\nu(t)} > 0,$$

so the coefficient of the x^{∇} term is in $\mathcal{R}^+_{\nu}$. Hence by Theorem 32 above, the equation can be written in self-adjoint form, with $p(t)$ and $q(t)$ in the desired form, and solutions

of equation (2.3) are also solutions of the self-adjoint form of the equation. □

Corollary 34. *If r_2 is ld-continuous and $(r_1 - \nu r_2) \in \mathcal{R}_\nu^+$, then the dynamic equation (2.4) can be written in self-adjoint form, with*

$$p(t) = \hat{e}_{(r_1 - \nu r_2)}(t, t_0) \quad \text{and} \quad q(t) = r_2(t)\hat{e}_{(r_1 - \nu r_2)}(t, t_0).$$

Furthermore, if x is a solution of (2.4), then x is also a solution of the self-adjoint form of the equation.

Proof. Suppose we have

$$x^{\Delta\nabla} + r_1(t)x^\nabla + r_2(t)x^\rho = 0.$$

Then

$$x^{\Delta\nabla} + r_1(t)x^\nabla + r_2(t)(x - \nu(t)x^\nabla) = 0,$$

or

$$x^{\Delta\nabla} + (r_1(t) - \nu(t)r_2(t))x^\nabla + r_2(t)x = 0.$$

This is in the form (2.2), and the coefficients meet the requirements of Theorem 32. Thus the result follows. □

2.3 Abel's Formula and Reduction of Order

We begin this section by looking at the Lagrange Identity for the dynamic equation (2.1). We establish several corollaries and related results, including Abel's Formula and its converse. We conclude the section with a reduction of order theorem. Some of the results in this section are due to Atici and Guseinov. Specifically, Theorems 35 and 43, and Corollaries 39 and 42 were previously established in their work [4]. Our conditions on p and q are less restrictive than Atici and Guseinov's, and our domain of interest, $\mathbb{D}$, is defined more broadly. In spite of this, however, many of the proofs contained in [4] remain valid. As this is the case, we have omitted the proofs of some of the following theorems, and refer the reader to Atici and Guseinov's work.

Theorem 35. *If $t_0 \in \mathbb{T}$, and x_0 and x_1 are given constants, then the initial value problem*

$$Lx = 0, \quad x(t_0) = x_0, \quad x^\Delta(t_0) = x_1$$

has a unique solution, and this solution exists on all of $\mathbb{T}$.

Definition 36. If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the *Wronskian* of x and y, denoted $W(x,y)(t)$ is defined by

$$W(x,y)(t) = \begin{vmatrix} x(t) & y(t) \\ x^\Delta(t) & y^\Delta(t) \end{vmatrix} \quad \text{for } t \in \mathbb{T}^\kappa.$$

Definition 37. If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the *Lagrange bracket* of x and y is defined by

$$\{x; y\}(t) = p(t)W(x, y)(t) \quad \text{for } t \in \mathbb{T}^\kappa.$$

Theorem 38 (Lagrange Identity). *If $x, y \in \mathbb{D}$, then*

$$x(t)Ly(t) - y(t)Lx(t) = \{x; y\}^\nabla(t) \quad \textit{for } t \in \mathbb{T}^\kappa_\kappa.$$

Proof. Let $x, y \in \mathbb{D}$. We have

$$\begin{aligned}
\{x; y\}^\nabla &= [pW(x, y)]^\nabla \\
&= [xpy^\Delta - ypx^\Delta]^\nabla \\
&= x^\nabla p^\rho y^{\Delta\rho} + x[py^\Delta]^\nabla - y^\nabla p^\rho x^{\Delta\rho} - y[px^\Delta]^\nabla \\
&= x^\nabla p^\rho y^\nabla + x[py^\Delta]^\nabla - y^\nabla p^\rho x^\nabla - y[px^\Delta]^\nabla \\
&= x[py^\Delta]^\nabla - y[px^\Delta]^\nabla \\
&= x([py^\Delta]^\nabla + qy) - y([px^\Delta]^\nabla + qx) \\
&= xLy - yLx,
\end{aligned}$$

where we have made use of the fact that x^Δ and y^Δ are continuous and applied Corollary 28. ☐

Corollary 39 (Abel's Formula). *If x, y are solutions of* (2.1), *then*

$$W(x, y)(t) = \frac{C}{p(t)} \quad \textit{for } t \in \mathbb{T}^\kappa,$$

where C is a constant.

Proof. If x, y are solutions of (2.1), they belong to $\mathbb{D}$. Then, by Theorem 38, we have

$$x(t)Ly(t) - y(t)Lx(t) = \{x; y\}^\nabla(t) \quad \text{for } t \in \mathbb{T}^\kappa.$$

But $Lx = Ly = 0$, so

$$0 = \{x; y\}^\nabla(t) \quad \text{for } t \in \mathbb{T}^\kappa.$$

Integrating, we see that

$$\{x; y\} = p(t)W(x, y)(t) = C,$$

which gives the desired result. ☐

Definition 40. Define the *inner product* of x and y on $[a, b]$ by

$$\langle x, y\rangle := \int_a^b x(t)y(t)\nabla t.$$

Corollary 41 (Green's Formula). *If $x, y \in \mathbb{D}$, then*

$$\langle x, Ly\rangle - \langle Lx, y\rangle = [p(t)W(x,y)]_a^b.$$

Proof. Integrating the expression in Theorem 38 gives the result immediately. □

Corollary 42. *If x, y are solutions of (2.1), then either*

(i) $W(x,y) \neq 0$ for $t \in \mathbb{T}^\kappa$ or

(ii) $W(x,y) \equiv 0$ for $t \in \mathbb{T}^\kappa$.

Case (i) occurs if and only if x and y are linearly independent on $\mathbb{T}$, and case (ii) occurs if and only if x and y are linearly dependent on $\mathbb{T}$.

In the standard way, one uses the uniqueness theorem to prove the following result.

Theorem 43. *If x_1 and x_2 are linearly independent solutions of (2.1) on $\mathbb{T}$, then a general solution of (2.1) is given by*

$$x(t) = c_1x_1(t) + c_2x_2(t).$$

Theorem 44 (Converse of Abel's Formula). *Assume u is a solution of (2.1) with $u(t) \neq 0$ for $t \in \mathbb{T}$. If $v \in \mathbb{D}$ satisfies*

$$W(u,v)(t) = \frac{C}{p(t)},$$

then v is also a solution of (2.1).

Proof. Suppose that u is a solution of (2.1) with $u(t) \neq 0$ for any t, and assume that $v \in \mathbb{D}$ satisfies $W(u,v)(t) = \frac{C}{p(t)}$. Then by Theorem 38, we have

$$u(t)Lv(t) - v(t)Lu(t) = \{u; v\}^\nabla(t),$$

so

$$\begin{aligned} u(t)Lv(t) &= [p(t)W(u,v)(t)]^\nabla \\ &= [p(t)\frac{C}{p(t)}]^\nabla \\ &= C^\nabla \\ &= 0. \end{aligned}$$

As $u(t) \neq 0$ for any t, we can divide through by it to get

$$Lv(t) = 0 \quad \text{for } t \in \mathbb{T}_\kappa^\kappa.$$

Hence v is a solution of (2.1) on $\mathbb{T}$. □

Theorem 45 (Reduction of Order). *Let $t_0 \in \mathbb{T}^\kappa$, and assume u is a solution of (2.1) with $u(t) \neq 0$ for any t. Then a second, linearly independent solution, v, of (2.1) is given by*

$$v(t) = u(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s$$

for $t \in \mathbb{T}$.

Proof. By Theorem 44, we need only show that $v \in \mathbb{D}$ and that $W(u,v)(t) = \frac{C}{p(t)}$ for some constant C. Consider first

$$\begin{aligned}
W(u,v)(t) &= \begin{vmatrix} u(t) & v(t) \\ u^\Delta(t) & v^\Delta(t) \end{vmatrix} \\
&= u(t)v^\Delta(t) - v(t)u^\Delta(t) \\
&= u(t)\left[u^\Delta(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s + \frac{u^\sigma(t)}{p(t)u(t)u^\sigma(t)}\right] \\
&\quad -u^\Delta(t)u(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s \\
&= u(t)u^\Delta(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s + \frac{u(t)u^\sigma(t)}{p(t)u(t)u^\sigma(t)} \\
&\quad -u(t)u^\Delta(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s \\
&= \frac{1}{p(t)}.
\end{aligned}$$

Here we have $C = 1$. It remains to show that $v \in \mathbb{D}$. We have that

$$\begin{aligned}
v^\Delta(t) &= u^\Delta(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s + \frac{u^\sigma(t)}{p(t)u(t)u^\sigma(t)} \\
&= u^\Delta(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s + \frac{1}{p(t)u(t)}.
\end{aligned}$$

Since $u \in \mathbb{D}, u(t) \neq 0$ and p is continuous, we have that v^Δ is continuous. Next, consider

$$\begin{aligned}
[p(t)v^\Delta(t)]^\nabla &= \left[p(t)u^\Delta(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s\right]^\nabla + \left[\frac{1}{u(t)}\right]^\nabla \\
&= [p(t)u^\Delta(t)]^\nabla \int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s \\
&\quad +p^\rho(t)u^{\Delta\rho}(t) \left[\int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s\right]^\nabla - \frac{u^\Delta(t)}{u(t)u^\rho(t)}.
\end{aligned}$$

Now, the first and last terms are ld-continuous. It is not as clear that the center term

is ld-continuous. Specifically, we are concerned about whether or not the expression

$$\left[\int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s\right]^\nabla$$

is ld-continuous. Note that the integrand is rd-continuous. Hence Theorem 27 applies and yields

$$\left[\int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s\right]^\nabla = \begin{cases} \frac{1}{p^\rho(t)u^\rho(t)u^{\sigma\rho}(t)} & t \in \mathbb{T}_A \\ \lim_{s\to t-} \frac{1}{p(s)u(s)u^\sigma(s)} & t \in A. \end{cases}$$

First, suppose $t \in \mathbb{T}_A$. Then $\sigma(\rho(t)) = t$, so our expression simplifies to

$$\left[\int_{t_0}^{t} \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s\right]^\nabla = \frac{1}{p^\rho(t)u^\rho(t)u^{\sigma\rho}(t)} = \frac{1}{p^\rho(t)u^\rho(t)u(t)}.$$

Next, suppose that $t \in A$. Then t is left-dense, and therefore,

$$\lim_{s\to t-} \sigma(s) = t.$$

Then as p and u are continuous, we have

$$\lim_{s\to t-} \frac{1}{p(s)u(s)u^\sigma(s)} = \frac{1}{p(t)u^2(t)}.$$

Now for $t \in A$, t is left-dense, so, if we like, we may write this expression as

$$\lim_{s\to t-} \frac{1}{p(s)u(s)u^\sigma(s)} = \frac{1}{p(t)u^2(t)} = \frac{1}{p^\rho(t)u^\rho(t)u(t)}.$$

This is the same expression we got for $t \in \mathbb{T}_A$, so we have that

$$\left[\int_{t_0}^{t} \frac{1}{p(\tau)u(\tau)u^\sigma(\tau)}\Delta \tau\right]^\nabla = \frac{1}{p^\rho(t)u^\rho(t)u(t)} \quad \text{for } t \in \mathbb{T}.$$

This function is ld-continuous, and so we have that $v \in \mathbb{D}$. Hence by Theorem 44, v is also a solution of (2.1). Finally, note that as $W(u,v)(t) = \frac{1}{p(t)} \neq 0$ for any t, u and v are linearly independent. $\square$

2.4 Oscillation and Disconjugacy

In this section, we establish results concerning generalized zeros of solutions of (2.1), and examine disconjugacy and oscillation of solutions.

Definition 46. We say that a solution, x, of (2.1) has a *generalized zero* at t if

$$x(t) = 0$$

or, if t is left-scattered and

$$x(\rho(t))x(t) < 0.$$

Definition 47. We say that (2.1) is *disconjugate* on an interval $[a, b]$ if the following hold.

(i) If x is a nontrivial solution of (2.1) with $x(a) = 0$, then x has no generalized zeros in $(a, b]$.

(ii) If x is a nontrivial solution of (2.1) with $x(a) \neq 0$, then x has at most one generalized zero in $(a, b]$.

Definition 48. Let $\omega = \sup \mathbb{T}$, and if $\omega < \infty$, assume $\rho(\omega) = \omega$. Let $a \in \mathbb{T}$. We say that (2.1) is *oscillatory* on $[a, \omega)$ if every nontrivial real-valued solution has infinitely many generalized zeros in $[a, \omega)$. We say (2.1) is *nonoscillatory* on $[a, \omega)$ if it is not oscillatory on $[a, \omega)$.

Lemma 49. *Let $\omega = \sup \mathbb{T}$. If $\omega < \infty$, then assume $\rho(\omega) = \omega$. Let $a \in \mathbb{T}$. Then if (2.1) is nonoscillatory on $[a, \omega)$, there is some $t_0 \in \mathbb{T}$, $t_0 \geq a$, such that (2.1) has a positive solution on $[t_0, \omega)$.*

Proof. Assume (2.1) is nonoscillatory on $[a, \omega)$, and then there is a nontrivial solution, u of (2.1) such that u has only finitely many generalized zeros in $[a, \omega)$. Let $b = \max\{t \in \mathbb{T} : u \text{ has a generalized zero at } t\}$. Fix $t_0 \in \mathbb{T}$ such that $t_0 > b$. Then either $u > 0$ on $[t_0, \omega)$ or $-u > 0$ on$[t_0, \omega)$. □

Theorem 50 (Sturm Separation Theorem). *Let u and v be linearly independent solutions of (2.1) on $\mathbb{T}$. Then u and v have no common zeros in $\mathbb{T}^{\kappa}$. If u has a zero at $t_1 \in \mathbb{T}$, and a generalized zero at $t_2 > t_1 \in \mathbb{T}$, then v has a generalized zero in $(t_1, t_2]$. If u has generalized zeros at $t_1 \in \mathbb{T}$ and $t_2 > t_1 \in \mathbb{T}$, then v has a generalized zero in $[t_1, t_2]$.*

Proof. If u and v have a common zero at $t_0 \in \mathbb{T}^{\kappa}$, then

$$W(u, v)(t_0) = \begin{vmatrix} u(t_0) & v(t_0) \\ u^{\Delta}(t_0) & v^{\Delta}(t_0) \end{vmatrix} = 0.$$

Hence u and v are linearly dependent.

Now suppose u has a zero at $t_1 \in \mathbb{T}$, and a generalized zero at $t_2 > t_1 \in \mathbb{T}$. Without loss of generality, we may assume $t_2 > \sigma(t_1)$ is the first generalized zero to the right of t_1, $u(t) > 0$ on (t_1, t_2), and $u(t_2) \leq 0$. Assume v is a linearly independent solution of (2.1) with no generalized zero in $(t_1, t_2]$. Without loss of generality, $v(t) > 0$ on $[t_1, t_2]$.

Then on $[t_1, t_2]$,

$$\left(\frac{u}{v}\right)^{\Delta}(t) = \frac{v(t)u^{\Delta}(t) - u(t)v^{\Delta}(t)}{v(t)v^{\sigma}(t)} = \frac{C}{p(t)v(t)v^{\sigma}(t)},$$

which is of one sign on $[t_1, t_2)$. Thus $\frac{u}{v}$ is monotone on $[t_1, t_2]$. Fix $t_3 \in (t_1, t_2)$. Note that

$$\frac{u(t_1)}{v(t_1)} = 0, \text{ and } \frac{u(t_3)}{v(t_3)} > 0.$$

But

$$\frac{u(t_2)}{v(t_2)} \leq 0,$$

which contradicts the fact that $\frac{u}{v}$ is monotone on $[t_1, t_2]$. Hence v must have a generalized zero in $(t_1, t_2]$.

Finally, suppose u has generalized zeros at $t_1 \in \mathbb{T}$ and $t_2 > t_1 \in \mathbb{T}$. Assume $t_2 > \sigma(t_1)$ is the first generalized zero to the right of t_1. If $u(t_1) = 0$, we are in the previous case, so assume $u(t_1) \neq 0$. Then, as u has a generalized zero at t_1, we have that t_1 is left-scattered. Without loss of generality, we may assume $u(t) > 0$ on $[t_1, t_2)$, $u(\rho(t_1)) < 0$ and $u(t_2) \leq 0$. Assume v is a linearly independent solution of (2.1) with no generalized zero in $[t_1, t_2)$. Without loss of generality, $v(t) > 0$ on $[t_1, t_2]$, and $v(\rho(t_1)) > 0$. In a similar fashion to the previous case, we apply Abel's Formula to get that $\frac{u}{v}$ is monotone on $[\rho(t_1), t_2]$. But

$$\frac{u(\rho(t_1))}{v(\rho(t_1))} < 0, \quad \frac{u(t_1)}{v(t_1)} > 0, \quad \text{and} \quad \frac{u(t_2)}{v(t_2)} \leq 0,$$

which is a contradiction. Hence v must have a generalized zero in $[t_1, t_2]$. □

Theorem 51. *If* (2.1) *has a positive solution on an interval* $\mathcal{I} \subset \mathbb{T}$, *then* (2.1) *is disconjugate on* $\mathcal{I}$. *Conversely, if* $a, b \in \mathbb{T}^{\kappa}_{\kappa}$ *and* (2.1) *is disconjugate on* $[\rho(a), \sigma(b)] \subset \mathbb{T}$, *then* (2.1) *has a positive solution on* $[\rho(a), \sigma(b)]$.

Proof. Assume (2.1) has a positive solution, u on $\mathcal{I} \subset \mathbb{T}$. If (2.1) is not disconjugate on $\mathcal{I}$, then (2.1) has a nontrivial solution v with at least two generalized zeros in $\mathcal{I}$. Then, without loss of generality, there are t_1, t_2 in $\mathcal{I}$ such that

$$v(t_1) \leq 0, v(t_2) \leq 0, \text{ and } v(t) > 0 \text{ on } (t_1, t_2) \text{ with } (t_1, t_2) \neq \emptyset.$$

Note that

$$\begin{aligned} \left(\frac{v}{u}\right)^{\Delta}(t) &= \frac{u(t)v^{\Delta}(t) - v(t)u^{\Delta}(t)}{u(t)u^{\sigma}(t)} \\ &= \frac{W(u,v)(t)}{u(t)u^{\sigma}(t)} \end{aligned}$$

$$= \frac{C}{p(t)u(t)u^\sigma(t)}$$

is of one sign on $\mathcal{I}^\kappa$. Hence $\frac{v}{u}$ is monotone on $\mathcal{I}$. But

$$\left(\frac{v}{u}\right)(t_1) \leq 0, \left(\frac{v}{u}\right)(t) > 0, \text{ and } \left(\frac{v}{u}\right)(t_2) \leq 0.$$

This contradicts the fact that $\frac{v}{u}$ is monotone. Hence(2.1) is disconjugate on $\mathcal{I}$.

Conversely, suppose that (2.1) is disconjugate on the compact interval $[\rho(a), \sigma(b)]$. Let u, v be the solutions of (2.1) satisfying $u(\rho(a)) = 0, u^\Delta(\rho(a)) = 1$ and $v(\sigma(b)) = 0, v^\Delta(b) = -1$. Since (2.1) is disconjugate on $[\rho(a), \sigma(b)]$, we have that $u(t) > 0$ on $(\rho(a), \sigma(b)]$, and $v(t) > 0$ on $[\rho(a), \sigma(b))$. Then

$$x(t) = u(t) + v(t)$$

is the desired positive solution. □

Theorem 52 (Pólya Factorization). *If* (2.1) *has a positive solution, u, on an interval $\mathcal{I} \subset \mathbb{T}$, then for any $x \in \mathbb{D}$, we get the* Pólya Factorization

$$Lx = \alpha_1(t)\{\alpha_2[\alpha_1 x]^\Delta\}^\nabla(t) \text{ for } t \in \mathcal{I},$$

where

$$\alpha_1 := \frac{1}{u} > 0 \quad \text{on } \mathcal{I},$$

and

$$\alpha_2 := puu^\sigma > 0 \quad \text{on } \mathcal{I}.$$

Proof. Assume that u is a positive solution of (2.1) on $\mathcal{I}$, and let $x \in \mathbb{D}$. Then by the Lagrange Identity (Theorem 38),

$$\begin{aligned}
u(t)Lx(t) - x(t)Lu(t) &= \{u; x\}^\nabla(t) \\
u(t)Lx(t) &= \{u; x\}^\nabla(t) \\
Lx(t) &= \frac{1}{u(t)}\{u; x\}^\nabla(t) \\
&= \frac{1}{u(t)}\{pW(u, x)\}^\nabla(t) \\
&= \frac{1}{u(t)}\left\{puu^\sigma\left[\frac{x}{u}\right]^\Delta\right\}^\nabla(t) \\
&= \alpha_1(t)\{\alpha_2\,[\alpha_1 x]^\Delta\}^\nabla(t),
\end{aligned}$$

for $t \in \mathcal{I}$, where α_1 and α_2 are as described in the theorem. □

Theorem 53 (Trench Factorization). *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, assume $\rho(\omega) = \omega$. If* (2.1) *is nonoscillatory on $[a, \omega)$, then there is $t_0 \in \mathbb{T}$ such that for any $x \in \mathbb{D}$, we get the* Trench Factorization

$$Lx(t) = \beta_1(t)\{\beta_2[\beta_1 x]^{\Delta}\}^{\nabla}(t)$$

for $t \in [t_0, \omega)$, where $\beta_1, \beta_2 > 0$ on $[t_0, \omega)$, and

$$\int_{t_0}^{\omega} \frac{1}{\beta_2(t)} \Delta t = \infty.$$

Proof. Since (2.1) is nonoscillatory on $[a, \omega)$, (2.1) has a positive solution, u on $[t_0, \omega)$ for some $t_0 \in \mathbb{T}$. Then by Theorem 52, Lx has a Pólya factorization on $[t_0, \omega)$. Thus there are functions α_1 and α_2 such that

$$Lx(t) = \alpha_1(t)\{\alpha_2[\alpha_1 x]^{\Delta}\}^{\nabla}(t) \text{ for } t \in [t_0, \omega),$$

with

$$\alpha_1 = \frac{1}{u} \text{ and } \alpha_2 = puu^{\sigma}.$$

Now, if

$$\int_{t_0}^{\omega} \frac{1}{\alpha_2(t)} \Delta t = \infty,$$

then take $\beta_1(t) = \alpha_1(t)$, and $\beta_2(t) = \alpha_2(t)$, and we are done. Therefore, assume that

$$\int_{t_0}^{\omega} \frac{1}{\alpha_2(t)} \Delta t < \infty.$$

In this case, let

$$\beta_1(t) = \frac{\alpha_1(t)}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \text{ and } \beta_2(t) = \alpha_2(t) \int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s$$

for $t \in [t_0, \omega)$. Note that as $\alpha_1, \alpha_2 > 0$, we have $\beta_1, \beta_2 > 0$ as well. Also,

$$\begin{aligned}
\int_{t_0}^{\omega} \frac{1}{\beta_2(t)} \Delta t &= \lim_{b \to \omega, b \in \mathbb{T}} \int_{t_0}^{b} \frac{1}{\alpha_2(t) \int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \Delta t \\
&= \lim_{b \to \omega, b \in \mathbb{T}} \int_{t_0}^{b} \frac{\frac{1}{\alpha_2(s)}}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \Delta t \\
&= \lim_{b \to \omega, b \in \mathbb{T}} \int_{t_0}^{b} \left[\frac{1}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \right]^{\Delta} \Delta t
\end{aligned}$$

$$= \lim_{b\to\omega, b\in\mathbb{T}} \left[\frac{1}{\int_b^\omega \frac{1}{\alpha_2(s)}\Delta s}\right]$$
$$= \infty.$$

Now let $x \in \mathbb{D}$. Then

$$[\beta_1 x]^\Delta(t) = \left[\frac{\alpha_1(t)x(t)}{\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s}\right]^\Delta = \frac{\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s[\alpha_1(t)x(t)]^\Delta + \alpha_1(t)x(t)\frac{1}{\alpha_2(t)}}{\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s \int_{\sigma(t)}^\omega \frac{1}{\alpha_2(s)}\Delta s}$$

for $t \in [t_0, \omega)$. So we get

$$\beta_2(t)[\beta_1(t)x]^\Delta = \alpha_2(t)[\alpha_1(t)x(t)]^\Delta \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s + \alpha_1(t)x(t)$$

for $t \in [t_0, \omega)$. Taking the ∇ -derivative of both sides gives

$$\begin{aligned}\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\}^\nabla &= \{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\}^\nabla \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s \\ &\quad + \{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\}^\rho \left[\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s\right]^\nabla \\ &\quad + [\alpha_1(t)x(t)]^\nabla\end{aligned}$$

for $t \in [t_0, \omega)$. We now claim that the last two terms in this expression cancel. Looking only at these last two terms, put the expression back in terms of our positive solution u. We get

$$\begin{aligned}&\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\}^\rho \left[\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s\right]^\nabla + [\alpha_1(t)x(t)]^\nabla \\ &= [p(t)u(t)u^\sigma(t)]^\rho \left[\frac{x(t)}{u(t)}\right]^{\Delta\rho} \left[\int_t^\omega \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s\right]^\nabla + \left[\frac{x(t)}{u(t)}\right]^\nabla.\end{aligned}$$

Now consider two cases:

Case 1: $t \in \mathbb{T}_A$. Then Theorem 26 applies, and we get

$$\begin{aligned}&[p(t)u(t)u^\sigma(t)]^\rho \left[\frac{x(t)}{u(t)}\right]^{\Delta\rho} \left[\int_t^\omega \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s\right]^\nabla + \left[\frac{x(t)}{u(t)}\right]^\nabla \\ &= -\frac{p^\rho(t)u^\rho(t)u(t)\left[\frac{x(t)}{u(t)}\right]^\nabla}{p^\rho(t)u^\rho(t)u(t)} + \left[\frac{x(t)}{u(t)}\right]^\nabla \\ &= -\left[\frac{x(t)}{u(t)}\right]^\nabla + \left[\frac{x(t)}{u(t)}\right]^\nabla \\ &= 0.\end{aligned}$$

Case 2: $t \in A$. In this case we have that $\rho(t) = t$, and we get

$$\begin{aligned}
& [p(t)u(t)u^\sigma(t)]^\rho \left[\frac{x(t)}{u(t)}\right]^{\Delta\rho} \left[\int_t^\omega \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s\right]^\nabla + \left[\frac{x(t)}{u(t)}\right]^\nabla \\
= & -\frac{[p(t)u(t)u^\sigma(t)]\left[\frac{u(t)x^\Delta(t)-x(t)u^\Delta(t)}{u(t)u^\sigma(t)}\right]}{p(t)u^2(t)} + \frac{u(t)x^\nabla(t) - x(t)u^\nabla(t)}{u^2(t)} \\
= & -\frac{u(t)x^{\Delta\rho}(t) - x(t)u^{\Delta\rho}(t)}{u^2(t)} + \frac{u(t)x^\nabla(t) - x(t)u^\nabla(t)}{u^2(t)} \\
= & \frac{-u(t)x^\nabla(t) + x(t)u^\nabla(t) + u(t)x^\nabla(t) - x(t)u^\nabla(t)}{u^2(t)} \\
= & \ 0.
\end{aligned}$$

Here, we have made use of the fact that $x, u \in \mathbb{D}$, which gives us that $x^{\Delta\rho} = x^\nabla$ and $u^{\Delta\rho} = u^\nabla$.

In either case, the last two terms cancel, and we have that

$$\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\}^\nabla = \{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\}^\nabla \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s.$$

It then follows that

$$\beta_1(t)\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla = \alpha_1(t)\left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla = Lx(t),$$

for $t \in [t_0, \omega)$ and the proof is complete. □

Theorem 54 (Recessive and Dominant Solutions). *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, then we assume $\rho(\omega) = \omega$. If (2.1) is nonoscillatory on $[a, \omega)$, then there is a solution, u, called a recessive solution at ω, such that u is positive on $[t_0, \omega)$ for some $t_0 \in \mathbb{T}$, and if v is any second, linearly independent solution, called a dominant solution at ω, the following hold.*

(i) $\lim_{t\to\omega-} \frac{u(t)}{v(t)} = 0$

(ii) $\int_{t_0}^\omega \frac{1}{p(t)u(t)u^\sigma(t)}\Delta t = \infty$

(iii) $\int_b^\omega \frac{1}{p(t)v(t)v^\sigma(t)}\Delta t < \infty$ *for $b < \omega$, sufficiently close, and*

(iv) $\frac{p(t)v^\Delta(t)}{v(t)} > \frac{p(t)u^\Delta(t)}{u(t)}$ *for $t < \omega$, sufficiently close.*

The recessive solution, u, is unique, up to multiplication by a nonzero constant.

Proof. As (2.1) is nonoscillatory, by Theorem 53, there is a Trench Factorization:

$$Lx(t) = \beta_1(t)\{\beta_2[\beta_1 x]^\Delta\}^\nabla(t),$$

where $\beta_1, \beta_2 > 0$ on $[t_0, \omega)$, and

$$\int_{t_0}^{\omega} \frac{1}{\beta_2(t)} \Delta t = \infty.$$

Then if $u(t) = \frac{1}{\beta_1(t)}$, u is a positive solution of (2.1). Now, let

$$v_0(t) = \frac{1}{\beta_1(t)} \int_{t_0}^{\omega} \frac{1}{\beta_2(s)} \Delta s.$$

Then,

$$\begin{aligned}
Lv_0(t) &= \beta_1(t)\{\beta_2[\beta_1 v_0]^{\Delta}\}^{\nabla}(t) \\
&= \beta_1(t)\{\beta_2[\beta_1 \frac{1}{\beta_1} \int_{t_0}^{\omega} \frac{1}{\beta_2(s)} \Delta s]^{\Delta}\}^{\nabla}(t) \\
&= \beta_1(t)\{\beta_2[\int_{t_0}^{\omega} \frac{1}{\beta_2(s)} \Delta s]^{\Delta}\}^{\nabla}(t) \\
&= \beta_1(t)\{\beta_2(t) \frac{1}{\beta_2(t)}\}^{\nabla} \\
&= \beta_1(t)\{1\}^{\nabla} \\
&= 0.
\end{aligned}$$

So v_0 is a solution of (2.1). Note that

$$\lim_{t \to \omega-} \frac{u(t)}{v_0(t)} = \lim_{t_0 \to \omega} \frac{1}{\int_{t_0}^{\omega} \frac{1}{\beta_2(s)} \Delta s} = 0,$$

as $\int_{t_0}^{\omega} \frac{1}{\beta_2(t)} \Delta t = \infty$. Now

$$\left(\frac{v_0}{u}\right)^{\Delta}(t) = \frac{W(u, v_0)(t)}{u(t)u^{\sigma}(t)} = \frac{C}{p(t)u(t)u^{\sigma}(t)}$$

where C is a constant by Theorem 39. Note that $C \neq 0$, since u and v_0 are linearly independent. Integrating both sides of this last equation from t_0 to t, we get

$$\frac{v_0(t)}{u(t)} = \int_{t_0}^{t} \frac{C}{p(s)u(s)u^{\sigma}(s)} \Delta s.$$

Taking the limit as $t \to \omega$, we get

$$\lim_{t \to \omega} \frac{v_0(t)}{u(t)} = \int_{t_0}^{\omega} \frac{C}{p(s)u(s)u^{\sigma}(s)} \Delta s,$$

and we see that
$$\int_{t_0}^{\omega} \frac{C}{p(s)u(s)u^\sigma(s)}\Delta s = \infty,$$
as desired.

Now let v be any solution of (2.1) such that u and v are linearly independent. Then
$$v(t) = c_1 u(t) + c_2 v_0(t), \quad \text{where } c_2 \neq 0,$$
and
$$\begin{aligned}
\lim_{t\to\omega-} \frac{u(t)}{v(t)} &= \lim_{t\to\omega-} \frac{u(t)}{c_1 u(t) + c_2 v_0(t)} \\
&= \lim_{t\to\omega-} \frac{\frac{u(t)}{v_0(t)}}{c_1 \frac{u(t)}{v_0(t)} + c_2} \\
&= 0.
\end{aligned}$$

Now, let v be a fixed solution of (2.1) such that u and v are linearly independent. Choose $t_1 \in [t_0, \omega)$ such that $v(t)v^\sigma(t) > 0$ on $[t_1, \omega)$. Then for $t \in [t_1, \omega)$,
$$\left(\frac{u}{v}\right)^\Delta (t) = \left(\frac{vu^\Delta - uv^\Delta}{vv^\sigma}\right)(t) = \frac{W(v,u)(t)}{v(t)v^\sigma(t)} = \frac{C_1}{p(t)v(t)v^\sigma(t)},$$
where $C_1 \neq 0$. Integrating,
$$\frac{u(t)}{v(t)} - \frac{u(t_1)}{v(t_1)} = \int_{t_1}^{t} \frac{C_1}{p(s)v(s)v^\sigma(s)}\Delta s.$$
Letting $t \to \omega-$, we see that
$$-\frac{u(t_1)}{v(t_1)} = \int_{t_1}^{\omega} \frac{C_1}{p(s)v(s)v^\sigma(s)}\Delta s,$$
which implies that
$$\int_{t_1}^{\omega} \frac{1}{p(s)v(s)v^\sigma(s)}\Delta s < \infty.$$
Furthermore, for $t \in [t_1, \omega)$,
$$\frac{p(t)v^\Delta(t)}{v(t)} - \frac{p(t)u^\Delta(t)}{u(t)} = \frac{p(t)W(u,v)(t)}{u(t)v(t)} = \frac{C_2}{u(t)v(t)}, \quad \text{where } C_2 \neq 0.$$
It remains to show that $C_2 > 0$. We have
$$\lim_{t\to\omega} \frac{v(t)}{u(t)} = \infty,$$

and

$$\left(\frac{v}{u}\right)^{\Delta}(t) = \frac{W(u,v)(t)}{u(t)u^{\sigma}(t)} = \frac{C_2}{p(t)u(t)u^{\sigma}(t)},$$

which implies that $C_2 > 0$, as desired.

Finally, we need to establish uniqueness, up to multiplication by a nonzero constant. Let u_1 be a recessive solution of (2.1), and suppose u_2 is another recessive solution. If u_1 and u_2 were linearly independent, u_2 would be a dominant solution. Hence u_1 and u_2 must be linearly dependent, and we see that $u_2 = ku_2$ for some nonzero constant k. □

2.5 The Riccati Equation

Usually, linear dynamic equations are considerably easier to solve than nonlinear ones. In this section, we are going to discuss the relationship between a particular nonlinear equation, called the Riccati equation, and our self-adjoint equation. We will see that there is a correspondence between solutions of these two equations. The Riccati equation is defined by

$$Rz = 0, \quad \text{where } Rz(t) := z^{\nabla}(t) + q(t) + \frac{(z^{\rho}(t))^2}{p^{\rho}(t) + \nu(t)z^{\rho}(t)} \tag{2.5}$$

for $t \in \mathbb{T}^{\kappa}_{\kappa}$. Here we assume that $p : \mathbb{T} \to \mathbb{R}$ is continuous, $q : \mathbb{T} \to \mathbb{R}$ is ld-continuous and that

$$p(t) > 0 \text{ for all } t \in \mathbb{T}.$$

Define the set $\mathbb{D}_R$ to be the set of all functions $z : \mathbb{T}^{\kappa} \to \mathbb{R}$ such that $z^{\nabla} : \mathbb{T}^{\kappa}_{\kappa} \to \mathbb{R}$ is ld-continuous and such that $p^{\rho}(t) + \nu(t)z^{\rho}(t) > 0$ for any $t \in \mathbb{T}^{\kappa}_{\kappa}$. A function $z \in \mathbb{D}_R$ is said to be a solution of $Rz = 0$ on $\mathbb{T}^{\kappa}$ provided $Rz(t) = 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Then we have the following theorem:

Theorem 55. *Assume $x \in \mathbb{D}$ has no generalized zeros in $\mathbb{T}$, and z is defined by the Riccati substitution*

$$z(t) = \frac{p(t)x^{\Delta}(t)}{x(t)}, \tag{2.6}$$

for $t \in \mathbb{T}^{\kappa}$. Then $z \in \mathbb{D}_R$, and

$$Lx(t) = x(t)Rz(t)$$

for $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Proof. We first wish to show that $z \in \mathbb{D}_R$. We have by the quotient rule

$$z^{\nabla}(t) = \left[\frac{p(t)x^{\Delta}(t)}{x(t)}\right]^{\nabla} = \frac{x(t)[p(t)x^{\Delta}(t)]^{\nabla} - p(t)x^{\Delta}(t)x^{\nabla}(t)}{x(t)x^{\rho}(t)},$$

which is ld-continuous on $\mathbb{T}^\kappa_\kappa$, since $x \in \mathbb{D}$. Next, note that

$$\begin{aligned}
p^\rho(t) + \nu(t)z^\rho(t) &= p^\rho(t) + \nu(t)\frac{p^\rho(t)x^{\Delta\rho}(t)}{x^\rho(t)} \\
&= \frac{p^\rho(t)(x^\rho(t) + \nu(t)x^\nabla(t))}{x^\rho(t)} \\
&= \frac{p^\rho(t)x(t)}{x^\rho(t)} > 0
\end{aligned}$$

for all $t \in \mathbb{T}^\kappa_\kappa$, since x has no generalized zeros in $\mathbb{T}$. It remains to show that $x(t)Rz(t) = Lx(t)$ for $t \in \mathbb{T}^\kappa_\kappa$. Suppressing the arguments, we get

$$\begin{aligned}
xRz &= x\left[z^\nabla + q + \frac{(z^\rho)^2}{p^\rho + \nu z^\rho}\right] \\
&= x\left[\left(\frac{px^\Delta}{x}\right)^\nabla + q + \frac{1}{p^\rho + \nu\left(\frac{px^\Delta}{x}\right)^\rho}\left(\left(\frac{px^\Delta}{x}\right)^\rho\right)^2\right] \\
&= x\left[\frac{x(px^\Delta)^\nabla - px^\Delta x^\nabla}{xx^\rho} + q + \frac{x^\rho}{p^\rho(x^\rho + \nu x^{\Delta\rho})}\frac{(p^\rho)^2(x^{\Delta\rho})^2}{(x^\rho)^2}\right] \\
&= (px^\Delta)^\nabla\frac{x}{x^\rho} - \frac{px^\Delta x^\nabla}{x^\rho} + qx + \frac{xp^\rho(x^\nabla)^2}{x^\rho(x^\rho + \nu x^\nabla)} \\
&= (px^\Delta)^\nabla\left(1 + \frac{\nu x^\nabla}{x^\rho}\right) - \frac{px^\Delta x^\nabla}{x^\rho} + qx + \frac{xp^\rho(x^\nabla)^2}{x^\rho x} \\
&= (px^\Delta)^\nabla + qx + \frac{(px^\Delta)^\nabla \nu x^\nabla}{x^\rho} - \frac{px^\Delta x^\nabla}{x^\rho} + \frac{p^\rho(x^\nabla)^2}{x^\rho} \\
&= Lx + \frac{(px^\Delta)^\nabla \nu x^\nabla - px^\Delta x^\nabla + p^\rho(x^\nabla)(x^{\Delta\rho})}{x^\rho} \\
&= Lx + \frac{x^\nabla(p^\rho x^{\Delta\rho} + \nu(px^\Delta)^\nabla) - px^\Delta x^\nabla}{x^\rho} \\
&= Lx + \frac{x^\nabla(px^\Delta) - px^\Delta x^\nabla}{x^\rho} \\
&= Lx.
\end{aligned}$$

Again, we have made use of the fact that $x \in \mathbb{D}$ and applied Corollary 28. $\square$

Theorem 56. *The self-adjoint equation* (2.1) *has a positive solution on* $\mathbb{T}$ *if and only if the Riccati equation* (2.5) *has a solution* z *on* $\mathbb{T}^\kappa$.

Proof. First, assume that x is a positive solution of (2.1), and let z be defined by the Riccati substitution (2.6). Then by Theorem 55, $z \in \mathbb{D}_R$, and $Lx = xRz$. Since x is a solution of $Lx = 0$ and has no generalized zeros, it follows that $Rz = 0$, as desired.

Conversely, assume that z is a solution of the Riccati equation, (2.5) on $\mathbb{T}^\kappa$. Then $z \in \mathbb{D}_R$, so $p^\rho(t) + \nu(t)z^\rho(t) > 0$ for all $t \in \mathbb{T}^\kappa_\kappa$, and z is continuous on $\mathbb{T}^\kappa$. This gives

us that $-\left(\frac{z}{p}\right)^{\rho} \in \mathcal{R}_{\nu}^{+}$, and thus, by Lemma 30, $\frac{z}{p} \in \mathcal{R}^{+}$. Now, let $t_0 \in \mathbb{T}$, and let x be the solution of the initial value problem

$$x^{\Delta} = \frac{z(t)}{p(t)}x, \quad x(t_0) = 1.$$

Note that although $\frac{z}{p}$ is only defined on $\mathbb{T}^{\kappa}$, x is defined on $\mathbb{T}$. Furthermore, as $x(t) = e_{\frac{z}{p}}(t, t_0)$, x is continuous and positive on $\mathbb{T}$. Next, consider

$$\begin{aligned}
\left[p(t)x^{\Delta}(t)\right]^{\nabla} &= [z(t)x(t)]^{\nabla} \\
&= z^{\nabla}(t)x^{\rho}(t) + z(t)x^{\nabla}(t) \\
&= z^{\nabla}(t)x^{\rho}(t) + z(t)x^{\Delta\rho}(t),
\end{aligned}$$

which is ld-continuous on $\mathbb{T}_{\kappa}^{\kappa}$. Hence $x \in \mathbb{D}$. Moreover, we see that

$$z(t) = \frac{p(t)x^{\Delta}(t)}{x(t)},$$

so by Theorem 55 $Lx = xRz = 0$. Hence x is the desired positive solution of (2.1). $\square$

Now, define $\mathbb{A}$ to be the set of functions

$$\mathbb{A} := \{u \in C_{pld}^{1}([\rho(a), \sigma(b)], \mathbb{R}) : u(\rho(a)) = u(\sigma(b)) = 0\}.$$

Here, C_{pld}^{1} denotes the set of all continuous functions whose ∇-derivatives are piecewise ld-continuous. Then we define the quadratic functional $\mathcal{F}$ on $\mathbb{A}$, by

$$\mathcal{F}(u) := \int_{\rho(a)}^{\sigma(b)} \left[p^{\rho}(t)(u^{\nabla}(t))^2 - q(t)u^2(t)\right] \nabla t.$$

Definition 57. We say $\mathcal{F}$ is *positive definite* on $\mathbb{A}$ provided $\mathcal{F}(u) \geq 0$ for all $u \in \mathbb{A}$, and $\mathcal{F}(u) = 0$ if and only if $u = 0$.

Lemma 58 (Completing the Square). *Assume z is a solution of the Riccati equation* (2.5) *on $[\rho(a), b]$. Let $u \in \mathbb{A}$. Then for all $t \in [a, b]$, we have*

$$\begin{aligned}
(zu^2)^{\nabla}(t) &= p^{\rho}(t)(u^{\nabla}(t))^2 - q(t)u^2(t) \\
&\quad - \left[\frac{z^{\rho}(t)u(t)}{\sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}} - \sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}\,u^{\nabla}(t)\right]^2.
\end{aligned}$$

Proof. Let z be a solution of the Riccati equation (2.5) on $[\rho(a), b]$, and let $u \in \mathbb{A}$. Then for $t \in [a, b]$,

$$(z(t)u^2(t))^{\nabla} = z^{\nabla}(t)(u^2(t)) + z^{\rho}(t)(u^2(t))^{\nabla}$$

$$
\begin{aligned}
&= z(t)^{\nabla}(u^2(t)) + z^{\rho}(t)(u^{\rho}(t)u^{\nabla}(t) + u(t)u^{\nabla}(t)) \\
&= \left[-q(t) - \frac{(z^{\rho}(t))^2}{p^{\rho}(t) + \nu(t)z^{\rho}(t)}\right] u^2(t) \\
&\quad + z^{\rho}(t)u^{\rho}(t)u^{\nabla}(t) + z^{\rho}(t)u(t)u^{\nabla}(t) \\
&= -q(t)u^2(t) - \frac{(z^{\rho}(t))^2 u^2(t)}{p^{\rho}(t) + \nu(t)z^{\rho}(t)} \\
&\quad + z^{\rho}(t)u(t)u^{\nabla}(t) + z^{\rho}(t)u^{\nabla}(t)(u(t) - \nu(t)u^{\nabla}(t)) \\
&= -q(t)u^2(t) - \frac{(z^{\rho}(t))^2 u^2(t)}{p^{\rho}(t) + \nu(t)z^{\rho}(t)} \\
&\quad + 2z^{\rho}(t)u(t)u^{\nabla}(t) - z^{\rho}(t)\nu(t)(u^{\nabla}(t))^2 \\
&= p^{\rho}(t)(u^{\nabla}(t))^2 - q(t)u^2(t) - \frac{(z^{\rho}(t))^2 u^2(t)}{p^{\rho}(t) + \nu(t)z^{\rho}(t)} \\
&\quad + 2z^{\rho}(t)u(t)u^{\nabla}(t) - (p^{\rho}(t) + z^{\rho}(t)\nu(t))(u^{\nabla}(t))^2 \\
&= p^{\rho}(t)(u^{\nabla}(t))^2 - q(t)u^2(t) \\
&\quad - \left[\frac{z^{\rho}(t)u(t)}{\sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}} - \sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}\,u^{\nabla}(t)\right]^2 .
\end{aligned}
$$

□

Theorem 59. *Let x be a solution of (2.1) on $[\rho(a), \sigma(b)]$, and let c, $d \in [\rho(a), \sigma(b)]$ with $\rho(a) \le c \le \sigma(c) < d \le \sigma(b)$. If $c = \rho(a)$, assume $x(c) = 0$. Now, let*

$$
u(t) = \begin{cases} 0 & \rho(a) \le t < c \\ x(t) & c \le t < d \\ 0 & d \le t \le \sigma(b). \end{cases}
$$

Then $u \in \mathbb{A}$, and $\mathcal{F}u = C + D$, where

$$
C = \begin{cases} -p(c)x^{\Delta}(c)x(c) & \nu(c) = 0 \\ \frac{p^{\rho}(c)x(c)x^{\rho}(c)}{\nu(c)} & \nu(c) > 0, \end{cases}
$$

$$
D = \begin{cases} p(d)x^{\Delta}(d)x(d) & \nu(d) = 0 \\ \frac{p^{\rho}(d)x(d)x^{\rho}(d)}{\nu(d)} & \nu(d) > 0. \end{cases}
$$

Proof. Let x, u be as described in the statement of the theorem. We first claim that $u \in \mathbb{A}$. It is apparent from the definition that $u \in C^1_{pld}([\rho(a), \sigma(b)], \mathbb{R})$, and that $u(\sigma(b)) = 0$. The fact that $u(\rho(a)) = 0$ is also clear from the definition unless $\rho(a) = c$. In this case, however, $u(\rho(a)) = u(c) = x(c) = 0$, by our assumption on x. So, $u \in \mathbb{A}$, as desired. Now consider

$$
\mathcal{F}u = \int_{\rho(a)}^{\sigma(b)} \left[p^{\rho}(t)(u^{\nabla}(t))^2 - q(t)u^2(t)\right] \nabla t.
$$

We have $u(t) = 0$ on $[\rho(a), c) \cup [d, \sigma(b)]$, and $u^\nabla = 0$ on $[\rho(a), c) \cup (d, \sigma(b)]$, so we get

$$\mathcal{F}u = \int_{\rho(c)}^{d} \left[p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t)\right] \nabla t.$$

Breaking up the integral, we get

$$\begin{aligned}
\mathcal{F}u &= \int_{\rho(c)}^{c} p^\rho(t)(u^\nabla(t))^2 \nabla t + \int_{c}^{\rho(d)} p^\rho(t)(u^\nabla(t))^2 \nabla t \\
&\quad + \int_{\rho(d)}^{d} p^\rho(t)(u^\nabla(t))^2 \nabla t - \int_{\rho(c)}^{c} q(t)u^2(t) \nabla t \\
&\quad - \int_{c}^{\rho(d)} q(t)u^2(t) \nabla t - \int_{\rho(d)}^{d} q(t)u^2(t) \nabla t.
\end{aligned}$$

Now, we apply Lemma 29 to get

$$\begin{aligned}
\mathcal{F}u &= p^\rho(c)(u^\nabla(c))^2\nu(c) + p^\rho(d)(u^\nabla(d))^2\nu(d) \\
&\quad -q(c)u^2(c)\nu(c) - q(d)u^2(d)\nu(d) \\
&\quad + \int_{c}^{\rho(d)} p^\rho(t)(u^\nabla(t))^2 \nabla t - \int_{c}^{\rho(d)} q(t)u^2(t) \nabla t.
\end{aligned}$$

Since $u(d) = 0$, the fourth term in this expression vanishes. Furthermore, $u(t) = x(t)$ on $[c, d)$, and $u^\nabla(t) = x^\nabla(t)$ on (c, d), thus we may substitute x for u in the two remaining integrals. We make this substitution and then evaluate the first of the two remaining integrals by parts, which yields

$$\begin{aligned}
\mathcal{F}u &= p^\rho(c)(u^\nabla(c))^2\nu(c) + p^\rho(d)(u^\nabla(d))^2\nu(d) \\
&\quad -q(c)u^2(c)\nu(c) - q(d)u^2(d)\nu(d) \\
&\quad + \int_{c}^{\rho(d)} p^\rho(t)(x^\nabla(t))^2 \nabla t - \int_{c}^{\rho(d)} q(t)x^2(t) \nabla t \\
&= p^\rho(c)(u^\nabla(c))^2\nu(c) + p^\rho(d)(u^\nabla(d))^2\nu(d) \\
&\quad -q(c)u^2(c)\nu(c) + p(\rho(d))x^\Delta(\rho(d))x(\rho(d)) \\
&\quad -p(c)x^\Delta(c)x(c) - \int_{c}^{\rho(d)} \left[p(t)x^\Delta(t)\right]^\nabla x(t) \nabla t \\
&\quad + \int_{c}^{\rho(d)} q(t)(x(t))^2 \nabla t \\
&= p^\rho(c)(u^\nabla(c))^2\nu(c) + p^\rho(d)(u^\nabla(d))^2\nu(d) \\
&\quad -q(c)u^2(c)\nu(c) + p(\rho(d))x^\Delta(\rho(d))x(\rho(d)) \\
&\quad -p(c)x^\Delta(c)x(c) - \int_{c}^{\rho(d)} x(t)Lx(t) \nabla t
\end{aligned}$$

$$\begin{aligned}
&= p^\rho(c)(u^\nabla(c))^2\nu(c) + p^\rho(d)(u^\nabla(d))^2\nu(d) \\
&\quad -q(c)u^2(c)\nu(c) + p(\rho(d))x^\Delta(\rho(d))x(\rho(d)) \\
&\quad -p(c)x^\Delta(c)x(c) \\
&= C + D,
\end{aligned}$$

where

$$\begin{aligned}
C &= \nu(c)p^\rho(c)(u^\nabla(c))^2 - \nu(c)q(c)(u(c))^2 \\
&\quad -p(c)x^\Delta(c)x(c),
\end{aligned}$$

and

$$D = \nu(d)p^\rho(d)(u^\nabla(d))^2 + p(\rho(d))x^\Delta(\rho(d))x(\rho(d)).$$

Note that if $\nu(c) = 0$, then $C = -p(c)x^\Delta(c)x(c)$. If $\nu(c) > 0$, then c is left-scattered, so we get

$$\begin{aligned}
C &= \nu(c)p^\rho(c)\left[\frac{u(c) - u(\rho(c))}{\nu(c)}\right]^2 - \nu(c)q(c)u^2(c) \\
&\quad -p(c)x^\Delta(c)x(c) \\
&= \frac{p^\rho(c)x^2(c)}{\nu(c)} - \nu(c)q(c)x^2(c) - p(c)x^\Delta(c)x(c) \\
&\quad +p^\rho(c)x^{\Delta\rho}(c)x(c) - p^\rho(c)x^{\Delta\rho}(c)x(c) \\
&= \frac{p^\rho(c)x^2(c)}{\nu(c)} - p^\rho(c)x^{\Delta\rho}(c)x(c) \\
&\quad -\nu(c)x(c)\left[q(c)x(c) + \frac{p(c)x^\Delta(c) - p^\rho(c)x^{\Delta\rho}(c)}{\nu(c)}\right] \\
&= \frac{p^\rho(c)x^2(c)}{\nu(c)} - p^\rho(c)x^\nabla(c)x(c) \\
&\quad -\nu(c)x(c)\left(q(c)x(c) + [px^\Delta]^\nabla(c)\right) \\
&= \frac{p^\rho(c)x^2(c)}{\nu(c)} - p^\rho(c)x(c)\left(\frac{x(c) - x^\rho(c)}{\nu(c)}\right) \\
&= \frac{p^\rho(c)x^2(c) - p^\rho(c)x^2(c) + p^\rho(c)x(c)x^\rho(c)}{\nu(c)} \\
&= \frac{p^\rho(c)x(c)x^\rho(c)}{\nu(c)},
\end{aligned}$$

so C is as described in the statement of the theorem. Now note that if $\nu(d) = 0$, then $D = p(\rho(d))x^\Delta(\rho(d))x(\rho(d)) = p(d)x^\Delta(d)x(d)$. If $\nu(d) > 0$, then d is left-scattered, so we get

$$D = \nu(d)p^\rho(d)(u^\nabla(d))^2 + p^\rho(d)x^\nabla(d)x^\rho(d)$$

$$
\begin{aligned}
&= \nu(d)p^\rho(d)\left[\frac{u(d)-u(\rho(d))}{\nu(d)}\right]^2 + p^\rho(d)x^\rho(d)\frac{x(d)-x^\rho(d)}{\nu(d)} \\
&= \frac{p^\rho(d)(x^\rho(d))^2}{\nu(d)} + \frac{p^\rho(d)x^\rho(d)x(d)}{\nu(d)} - \frac{p^\rho(d)(x^\rho(d))^2}{\nu(d)} \\
&= \frac{p^\rho(d)x^\rho(d)x(d)}{\nu(d)}.
\end{aligned}
$$

Thus D is as desired, and the proof is complete. □

Theorem 60 (Jacobi's Condition). *The self-adjoint equation* (2.1) *is disconjugate on* $[\rho(a), \sigma(b)]$ *if and only if* $\mathcal{F}$ *is positive definite on* $\mathbb{A}$.

Proof. First, suppose (2.1) is disconjugate on $[\rho(a), \sigma(b)]$. Then there is a positive solution, x, of (2.1) on $[\rho(a), \sigma(b)]$. Let $z(t) := \frac{p(t)x^\Delta(t)}{x(t)}$. Then by Theorem 56, z is a solution of $Rz = 0$ on $[\rho(a), b]$. Thus by Lemma 58, for any $u \in \mathbb{A}$,

$$
\begin{aligned}
(z(t)u^2(t))^\nabla &= p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t) \\
&\quad - \left[\frac{z^\rho(t)u(t)}{\sqrt{p^\rho(t)+\nu(t)z^\rho(t)}} - \sqrt{p^\rho(t)+\nu(t)z^\rho(t)}\,u^\nabla(t)\right]^2
\end{aligned}
$$

for $t \in [a, b]$. In fact, it can be shown that this equation holds at $t = \sigma(b)$ as well. As the equation holds on $[a, \sigma(b)]$, we may integrate from $\rho(a)$ to $\sigma(b)$, and noting that $u(\rho(a)) = u(\sigma(b)) = 0$, we get

$$
\mathcal{F}u = \int_{\rho(a)}^{\sigma(b)} \left[\frac{z^\rho(t)u(t)}{\sqrt{p^\rho(t)+\nu(t)z^\rho(t)}} - \sqrt{p^\rho(t)+\nu(t)z^\rho(t)}\,u^\nabla(t)\right]^2 \nabla t,
$$

so $\mathcal{F}u \geq 0$ for all $u \in \mathbb{A}$. Furthermore, it is clear that if $u \equiv 0$, then $\mathcal{F}u = 0$. Now suppose $\mathcal{F}u = 0$. Then

$$
\frac{z^\rho(t)u(t)}{\sqrt{p^\rho(t)+\nu(t)z^\rho(t)}} = \sqrt{p^\rho(t)+\nu(t)z^\rho(t)}\,u^\nabla(t),
$$

so u solves the initial value problem

$$
u^\nabla = \frac{z^\rho}{p^\rho + \nu z^\rho}\,u, \quad u(\sigma(b)) = 0
$$

on $[a, \sigma(b)]$ Since $\frac{z^\rho}{p^\rho+\nu z^\rho} \in \mathcal{R}_\nu^+$, the solution of this IVP is unique, and gives $u(t) \equiv 0$ on $[a, \sigma(b)]$. As $u(\rho(a)) = 0$ as well, we get $u(t) \equiv 0$ on $[\rho(a), \sigma(b)]$. Hence, $\mathcal{F}$ is positive definite on $\mathbb{A}$.

We will prove the converse of this statement by contrapositive. Suppose (2.1) is not disconjugate on $[\rho(a), \sigma(b)]$. Then there is a nontrivial solution x of (2.1)

such that either $x(\rho(a)) = 0$ and x has a generalized zero in $(\rho(a), \sigma(b)]$, or x has two generalized zeros in $(\rho(a), \sigma(b)]$. In either case, let $c < d$ be the two smallest generalized zeros of x in $[\rho(a), \sigma(b)]$. Then, let

$$u(t) = \begin{cases} 0 & \rho(a) \le t < c \\ x(t) & c \le t < d \\ 0 & d \le t \le \sigma(b). \end{cases}$$

Applying Theorem 59, we then have $\mathcal{F} = C + D \le 0$. As u is not identically 0, this tells us that $\mathcal{F}$ is not positive definite. By contrapositive, the proof is complete. $\square$

Theorem 61 (Sturm Comparison Theorem). *Let*

$$L_1x = [p_1(t)x^{\Delta}]^{\nabla} + q_1(t)x,$$

$$L_2x = [p_2(t)x^{\Delta}]^{\nabla} + q_2(t)x.$$

Assume $q_1(t) \ge q_2(t)$ and $0 < p_1(t) \le p_2(t)$ for $t \in [\rho(a), \sigma(b)]$. If $L_1x(t) = 0$ is disconjugate on $[\rho(a), \sigma(b)]$, then $L_2x(t) = 0$ is disconjugate on $[\rho(a), \sigma(b)]$.

Proof. Let

$$\mathcal{F}_1(u) := \int_{\rho(a)}^{\sigma(b)} \left[p_1^{\rho}(t)(u^{\nabla}(t))^2 - q_1(t)u^2(t)\right] \nabla t,$$

$$\mathcal{F}_2(u) := \int_{\rho(a)}^{\sigma(b)} \left[p_2^{\rho}(t)(u^{\nabla}(t))^2 - q_2(t)u^2(t)\right] \nabla t.$$

Assume that $L_1x(t) = 0$ is disconjugate on $[\rho(a), \sigma(b)]$. Then by Theorem 60, the quadratic functional $\mathcal{F}_1$ is positive definite on $\mathbb{A}$. Then, for $u \in \mathbb{A}$, we have

$$\begin{aligned} \mathcal{F}_2 u &= \int_{\rho(a)}^{\sigma(b)} \left[p_2^{\rho}(t)(u^{\nabla}(t))^2 - q_2(t)u^2(t)\right] \nabla t \\ &\ge \int_{\rho(a)}^{\sigma(b)} \left[p_1^{\rho}(t)(u^{\nabla}(t))^2 - q_1(t)u^2(t)\right] \nabla t \\ &= \mathcal{F}_1 u. \end{aligned}$$

Hence $\mathcal{F}_2$ is positive definite on $\mathbb{A}$. Then, again by Theorem 60, $L_2x = 0$ is disconjugate on $[\rho(a), \sigma(b)]$. $\square$

Theorem 62. *Let L_1x, L_2x be as in the Sturm Comparison Theorem. Then if $L_ix = 0$ is disconjugate on $[\rho(a), \sigma(b)]$ for $i = 1, 2$, and if*

$$p(t) = \lambda_1 p_1(t) + \lambda_2 p_2(t), \text{ and}$$

$$q(t) = \lambda_1 q_1(t) + \lambda_2 q_2(t),$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, then $Lx = 0$ is disconjugate on $[\rho(a), \sigma(b)]$.

Proof. Suppose $L_i x = 0$ is disconjugate on $[\rho(a), \sigma(b)]$ for $i = 1, 2$. Then the quadratic functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ are positive definite on $\mathbb{A}$. Then for $u \in \mathbb{A}$, we have

$$\begin{aligned}
\mathcal{F}u &= \int_{\rho(a)}^{\sigma(b)} \left[p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t)\right] \nabla t \\
&= \int_{\rho(a)}^{\sigma(b)} \left[(\lambda_1 p_1(t) + \lambda_2 p_2(t))(u^\nabla(t))^2 - (\lambda_1 q_1(t) + \lambda_2 q_2(t))u^2(t)\right] \nabla t \\
&= \int_{\rho(a)}^{\sigma(b)} \left[p_1^\rho(t)(u^\nabla(t))^2 - q_1(t)u^2(t)\right] \nabla t \\
&\quad + \int_{\rho(a)}^{\sigma(b)} \left[p_2^\rho(t)(u^\nabla(t))^2 - q_2(t)u^2(t)\right] \nabla t \\
&= \mathcal{F}_1 u + \mathcal{F}_2 u.
\end{aligned}$$

Therefore, $\mathcal{F}$ is positive definite on $\mathbb{A}$, and hence $Lx = 0$ is disconjugate on $[\rho(a), \sigma(b)]$. □

We summarize some of the major results in the following theorem.

Theorem 63 (Reid Roundabout Theorem). *The following are equivalent:*

(i) $Lx = 0$ *is disconjugate on* $[\rho(a), \sigma(b)]$*;*

(ii) $Lx = 0$ *has a positive solution on* $[\rho(a), \sigma(b)]$*;*

(iii) The quadratic functional $\mathcal{F}$ *is positive definite on* $\mathbb{A}$*;*

(iv) the Riccati differential inequality $Rz \leq 0$ *has a solution on* $[\rho(a), b]$.

Proof. By Theorem 51, (i) and (ii) are equivalent. By Theorem 60, (i) and (iii) are equivalent. By Theorem 56, (ii) implies (iv). It remains to show that (iv) implies (i). So, assume $Rz \leq 0$ has a solution, z on $[\rho(a), b]$, and let

$$w(t) := Rz(t) \quad \text{for} \quad t \in [a, b].$$

If $\rho(a) < a$, let $w(\rho(a)) = 0$, and if $\sigma(b) > b$, let $w(\sigma(b)) = 0$. Then z is a solution of the Riccati equation

$$z^\nabla(t) + (q(t) - w(t)) + \frac{(z^\rho(t))^2}{p^\rho(t) + \nu(t) z^\rho(t)} = 0$$

on $[\rho(a), b]$. This implies that the self-adjoint dynamic equation

$$(p(t)x^\Delta)^\nabla + (q(t) - w(t))x = 0$$

is disconjugate on $[\rho(a), \sigma(b)]$. But

$$q(t) - w(t) \geq q(t)$$

on $[\rho(a), \sigma(b)]$, so by Theorem 61, $Lx = 0$ is disconjugate on $[\rho(a), \sigma(b)]$, and the proof is complete. $\square$

Theorem 64 (Wintner's Theorem). *Assume* $\sup \mathbb{T} = \infty$, $a \in \mathbb{T}$, $\nu(t) \geq K > 0$, *and* $0 < p(t) \leq M$ *for some constants* K *and* M. *If*

$$\int_a^\infty q(t)\ \nabla t = \infty,$$

then (2.1) *is oscillatory.*

Proof. By way of contradiction assume the hypotheses of the theorem hold, but (2.1) is nonoscillatory. Then by Lemma 49 there is some $t_0 \in \mathbb{T}$ such that (2.1) has a positive solution on $[t_0, \infty)$. Define $z : \mathbb{T} \to \mathbb{R}$ by

$$z(t) := \frac{p(t)x^\Delta(t)}{x(t)}.$$

Then by Theorem 56, z is a solution of the Riccati equation $Rz = 0$, and $z \in \mathbb{D}_R$, so $p^\rho(t) + \nu(t)z^\rho(t) > 0$, and therefore,

$$z^\rho(t) > -\frac{p^\rho(t)}{\nu(t)} \geq -\frac{M}{K},$$

If we integrate both sides of the Riccati equation, we see that

$$\begin{aligned} z(t) - z(t_0) &= -\int_{t_0}^\infty q(t)\ \nabla t - \int_{t_0}^\infty \frac{(z^\rho(t))^2}{p^\rho(t) + \nu(t)z^\rho(t)}\ \nabla t \\ &\leq -\int_{t_0}^\infty q(t)\ \nabla t. \end{aligned}$$

This implies that $\lim_{t\to\infty} z(t) = -\infty$, which is a contradiction. $\square$

Theorem 64 is a special case of the Leighton-Wintner Theorem, which we now state.

Theorem 65 (Leighton-Wintner Theorem). *Assume* $\sup \mathbb{T} = \infty$, $a \in \mathbb{T}$, *and* $p(t) > 0$ *for all* $t \in \mathbb{T}$. *If*

$$\int_a^\infty \frac{1}{p(t)}\ \Delta t = \int_a^\infty q(t)\ \nabla t = \infty,$$

then (2.1) *is oscillatory on* $[a, \infty)$.

Proof. By way of contradiction, assume that (2.1) is nonoscillatory. Then there is some $t_0 \in \mathbb{T}$ such that (2.1) is disconjugate on $[t_0, \infty)$. Then by Theorem 54, there is a dominant solution, $v(t)$ such that

$$\int_b^\infty \frac{1}{p(t)v(t)v^\sigma(t)}\,\Delta t < \infty$$

for $b \in \mathbb{T}$, sufficiently large. Without loss of generality, we may take $v(t) > 0$ on $[b, \infty)$.

Define $z(t)$ by the Riccati substitution. Then by similar reasoning to the proof of Theorem 64, we have

$$z(t) - z(b) \leq \int_b^\infty q(t)\nabla t,$$

so $\lim_{t\to\infty} z(t) = -\infty$. Then we can find $t_1 \in \mathbb{T}$ such that $t_1 > b$, and $z(t) < 0$ on $[t_1, \infty)$. Then as $z = p(t)v^\Delta(t)/v(t)$, v is positive and decreasing on $[t_1, \infty)$, so

$$\begin{aligned}\int_{t_1}^\infty \frac{1}{p(t)v(t)v^\sigma(t)}\,\Delta t &\geq \frac{1}{v^2(t_1)}\int_{t_1}^\infty \frac{1}{p(t)}\,\Delta t \\ &= \infty,\end{aligned}$$

which is a contradiction. $\square$

Chapter 3

Linear Dynamic Equations in Factored Form

3.1 Linear Equations with Delta-Derivatives

3.1.1 The General Case

In this section, we look at solution techniques for linear equations containing delta-derivatives which can be written in factored form. Fix $n \in \mathbb{N}$, and for $1 \le i \le n$ let $a_i : \mathbb{T} \to \mathbb{R}$ be rd-continuous. Furthermore, for $1 \le i \le n$, assume $a_i(t) \neq 0$ for any $t \in \mathbb{T}$. Define the operators D_{a_i}, $1 \le i \le n$ by

$$[D_{a_i} y](t) := a_i(t) y^{\Delta}(t).$$

For $1 \le i \le n$, we take the domain of D_{a_i} to be the family of all delta-differentiable functions. Clearly each D_{a_i} is a linear operator.

Now consider the dynamic equation

$$Ly = 0 \quad \text{where} \quad Ly = (D_{a_1} - \lambda_1)(D_{a_2} - \lambda_2) \ldots (D_{a_n} - \lambda_n) y. \tag{3.1}$$

Here λ_i, $1 \le i \le n$ are (possibly complex) constants. Note that the order in which the factors are written down is important, as the factors do not necessarily commute with one another. We define the domain of the operator L, which we will denote by $\mathbb{D}$, to be the set of all functions, $y : \mathbb{T} \to \mathbb{R}$, such that $(D_{a_2} - \lambda_2) \ldots (D_{a_n} - \lambda_n) y : \mathbb{T}^{\kappa^{n-1}} \to \mathbb{R}$ is defined, and is delta-differentiable. In the case $n = 1$, we understand this to mean that our domain is just the set of all delta-differentiable functions. We say that $y \in \mathbb{D}$ is a solution of (3.1) provided $Ly(t) = 0$ for all $t \in \mathbb{T}^{\kappa^n}$.

Definition 66. We say the dynamic equation (3.1) is *regressive* provided $\frac{\lambda_i}{a_i} \in \mathcal{R}$ for $1 \le i \le n$.

Theorem 67. *The dynamic equation* (3.1) *is equivalent to the system*

$$x^{\Delta} = A(t)x, \tag{3.2}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$A(t) = \begin{bmatrix} \frac{\lambda_n}{a_n(t)} & \frac{1}{a_n(t)} & 0 & \dots & 0 \\ 0 & \frac{\lambda_{n-1}}{a_{n-1}(t)} & \frac{1}{a_{n-1}(t)} & \dots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \frac{\lambda_2}{a_2(t)} & \frac{1}{a_2(t)} \\ 0 & \dots & \dots & 0 & \frac{\lambda_1}{a_1(t)} \end{bmatrix}.$$

Proof. To see that this system is actually equivalent to our original equation, suppose y is a solution of (3.1), and let

$$\begin{aligned} x_1 &= y \\ x_2 &= (D_{a_n} - \lambda_n)y \\ &\vdots \\ x_n &= (D_{a_2} - \lambda_2)\dots(D_{a_n} - \lambda_n)y. \end{aligned}$$

In other words,

$$x_1 = y, \quad \text{and} \quad x_{i+1} = (D_{a_{n+1-i}} - \lambda_{n+1-i})x_i, \quad 1 \le i \le n-1.$$

Then for $1 \le i \le n-1$, we have

$$\begin{aligned} x_{i+1} &= (D_{a_{n+1-i}} - \lambda_{n+1-i})x_i \\ &= a_{n+1-i}x_i^{\Delta} - \lambda_{n+1-i}x_i. \end{aligned}$$

Solving for x_i^{Δ}, we see that

$$x_i^{\Delta} = \frac{\lambda_{n+1-i}}{a_{n+1-i}}x_i + \frac{1}{a_{n+1-i}}x_{i+1}.$$

Furthermore, by our dynamic equation, we have

$$Ly = (D_{a_1} - \lambda_1)x_n = a_1 x_n^{\Delta} - \lambda_1 x_n = 0.$$

Solving for x_n^{Δ}, we get

$$x_n^{\Delta} = \frac{\lambda_1}{a_1} x_n,$$

and we see that x is a solution of (3.2).

Conversely, suppose x is a solution of (3.2). We claim that $y := x_1$ is a solution of (3.1). Note that

$$x_n^{\Delta} = \frac{\lambda_1}{a_1} x_n,$$

and thus

$$(D_{a_1} - \lambda_1)x_n = 0.$$

Now we also have that

$$x_{n-1}^{\Delta} = \frac{\lambda_2}{a_2} x_{n-1} + \frac{1}{a_2} x_n.$$

Solving this for x_n, we get

$$x_n = a_2(t)x_{n-1}^{\Delta} - \lambda_2 x_{n-1} = (D_{a_2} - \lambda_2)x_{n-1},$$

and therefore

$$(D_{a_1} - \lambda_1)(D_{a_2} - \lambda_2)x_{n-1} = 0.$$

Continuing in this fashion, we get

$$(D_{a_1} - \lambda_1)(D_{a_2} - \lambda_2)\dots(D_{a_n} - \lambda_n)x_1 = 0,$$

which is the desired result. □

Theorem 68. *If* (3.1) *is regressive,* $t_0 \in \mathbb{T}$, $y_i \in \mathbb{R}$, $1 \le i \le n$, *then the initial value problem*

$$Ly = 0,$$

$$y(t_0) = y_1, \quad (D_{a_n} - \lambda_n)y(t_0) = y_2, \quad \dots, \quad (D_{a_2} - \lambda_2)\dots(D_{a_n} - \lambda_n)y(t_0) = y_n \tag{3.3}$$

has a unique solution.

Proof. Suppose (3.1) is regressive, and consider the equivalent system (3.2). By [6, Theorem 5.8], it suffices to show that the matrix-valued function $A(t)$ is rd-continuous

and regressive. Recall that $A(t)$ was defined by

$$A(t) := \begin{bmatrix} \frac{\lambda_n}{a_n(t)} & \frac{1}{a_n(t)} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_{n-1}}{a_{n-1}(t)} & \frac{1}{a_{n-1}(t)} & \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \frac{\lambda_2}{a_2(t)} & \frac{1}{a_2(t)} \\ 0 & \cdots & \cdots & 0 & \frac{\lambda_1}{a_1(t)} \end{bmatrix}.$$

Then, as each function, a_i, $1 \leq i \leq n$, was assumed to be rd-continuous, we have that $A(t)$ is rd-continuous as well. To show $A(t)$ is regressive, we must show that $\det(I + \mu(t)A(t)) \neq 0$ for any $t \in \mathbb{T}$. We get

$$\begin{aligned} & \det(I + \mu(t)A(t)) \\ = \ & \det \begin{bmatrix} 1+\mu(t)\frac{\lambda_n}{a_n(t)} & \mu(t)\frac{1}{a_n(t)} & 0 & \cdots & 0 \\ 0 & 1+\mu(t)\frac{\lambda_{n-1}}{a_{n-1}(t)} & \mu(t)\frac{1}{a_{n-1}(t)} & \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1+\mu(t)\frac{\lambda_2}{a_2(t)} & \mu(t)\frac{1}{a_2(t)} \\ 0 & \cdots & \cdots & 0 & 1+\mu(t)\frac{\lambda_1}{a_1(t)} \end{bmatrix} \\ = \ & \prod_{i=1}^{n} \left(1 + \mu(t)\frac{\lambda_i}{a_i(t)}\right) \\ \neq \ & 0, \end{aligned}$$

since (3.1) is regressive. Thus the IVP (3.1), (3.3) has a unique solution. □

Now that we have established that solutions of IVPs exist and are unique, we turn our attention to actually *finding* these solutions, and examining their properties.

Definition 69. For $1 \leq i \leq n$, let $y_i \in \mathbb{D}$. We define the *Wronskian* associated with $Ly = 0$, $W = W(y_1, \ldots, y_n) : \mathbb{T}^{\kappa^{n-1}} \to \mathbb{R}$ by

$$W(y_1, \ldots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ D_{a_n}y_1 & D_{a_n}y_2 & \cdots & D_{a_n}y_n \\ D_{a_{n-1}}(D_{a_n}y_1) & D_{a_{n-1}}(D_{a_n}y_2) & \cdots & D_{a_{n-1}}(D_{a_n}y_n) \\ \vdots & \vdots & & \vdots \\ D_{a_2}(\ldots(D_{a_n}y_1)) & D_{a_2}(\ldots(D_{a_n}y_2)) & \cdots & D_{a_2}(\ldots(D_{a_n}y_n)) \end{bmatrix}.$$

Adding (nonzero) multiples of one row to another row does not change the deter-

minant of a matrix, so we may write the Wronskian in the equivalent form

$$W = \det \begin{bmatrix} y_1 & \dots & y_n \\ (D_{a_n} - \lambda_n)y_1 & \dots & (D_{a_n} - \lambda_n)y_n \\ \vdots & & \vdots \\ (D_{a_2} - \lambda_2)\dots(D_{a_n} - \lambda_n)y_1 & \dots & (D_{a_2} - \lambda_2)\dots(D_{a_n} - \lambda_n)y_n \end{bmatrix}.$$

We then get the following theorem.

Theorem 70. *If* (3.1) *is regressive, and* $y_1, \dots, y_n$ *are solutions of* (3.1), *then either*

(i) $W(y_1, \dots, y_n) \equiv 0$, *or*

(ii) $W(y_1, \dots, y_n)(t) \neq 0$ *for any* $t \in \mathbb{T}$.

The second case occurs if and only if the functions y_i, $1 \leq i \leq n$ *are linearly independent on* $\mathbb{T}$.

Proof. Suppose (3.1) is regressive and $y_1, \dots, y_n$ are solutions of (3.1). Define the n vector valued functions, $x_1, \dots, x_n$ by

$$x_i = \begin{bmatrix} y_i \\ (D_{a_n} - \lambda_n)y_i \\ \vdots \\ (D_{a_2} - \lambda_2)\dots(D_{a_n} - \lambda_n)y_i \end{bmatrix}.$$

Then we see that for $1 \leq i \leq n$, x_i is a solution of the system (3.2). Next, define the column matrix, X by

$$X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}.$$

Then X solves the matrix dynamic equation $X^\Delta = A(t)X$. Furthermore, we see that

$$W(y_1, \dots, y_n) = \det X.$$

Since (3.1) is regressive, $A(t)$ is regressive, and we may apply a theorem developed by Cormani in [8] to conclude that

$$W^\Delta = q(t)W, \tag{3.4}$$

where $q \in \mathcal{R}$. In her paper, Cormani describes the function q in detail. Here, however, we are only interested in the fact that $q \in \mathcal{R}$, and since the expression describing q is quite complicated, we choose to omit it.

Now, since $q \in \mathcal{R}$, we can solve equation (3.4), and we see that

$$W(t) = e_q(t, t_0)W(t_0).$$

The exponential function $e_q(t,t_0)$ is never zero, and we see that if $W(t_0)=0$, then $W\equiv 0$, and if $W(t_0)\neq 0$, then $W(t)\neq 0$ for any $t\in\mathbb{T}$.

It remains to show that $W(t)\neq 0$ for any $t\in\mathbb{T}$ if and only if $y_1,\dots,y_n$ are linearly independent.

Suppose

$$c_1y_1+c_2y_2+\ldots+c_ny_n=0,$$

where c_i, $1\le i\le n$ are constants. Then

$$\begin{aligned} c_1(D_{a_n}-\lambda_n)y_1+c_2(D_{a_n}-\lambda_n)y_2+\ldots+c_n(D_{a_n}-\lambda_n)y_n &= 0\\ &\vdots\\ c_1(D_{a_2}-\lambda_2)\ldots(D_{a_n}-\lambda_n)y_1+\ldots+c_n(D_{a_2}-\lambda_2)\ldots(D_{a_n}-\lambda_n)y_n &= 0, \end{aligned}$$

and we see that

$$c:=\begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_n \end{bmatrix}$$

satisfies

$$X(t)c=0,$$

where X was defined in the first part of this proof. This holds for any $t\in\mathbb{T}^{\kappa^{n-1}}$, so we may evaluate X at an arbitrary point, $t_1\in\mathbb{T}^{\kappa^{n-1}}$, and we get

$$X(t_1)c=0.$$

Now recall that $W(y_1,\dots,y_n)=\det X$. Hence if $W(t_1)\neq 0$, then $X(t_1)$ is invertible, and we see that $c=0$, which implies that $y_1,\dots,y_n$ are linearly independent. If, on the other hand, $W\equiv 0$, then $X(t_1)$ is not invertible, and the system $X(t_1)c=0$ has a nontrivial solution. In other words, there are constants, $\hat{c}_1,\dots,\hat{c}_n$, not all zero, such that

$$y:=\hat{c}_1y_1+\hat{c}_2y_2+\ldots+\hat{c}_ny_n$$

satisfies $y(t_1)=0$. Therefore, by the linearity of the operators D_{a_i}, $1\le i\le n$,

$$y(t_1)=(D_{a_n}-\lambda_n)y(t_1)=\ldots=(D_{a_2}-\lambda_2)\ldots(D_{a_n}-\lambda_n)y(t_1)=0.$$

Furthermore, since $y_1,\dots y_n$ are solutions of (3.1), y is also a solution of (3.1). By Theorem 68, however, solutions of initial value problems are unique, and we see that this implies

$$y=\hat{c}_1y_1+\hat{c}_2y_2+\ldots+\hat{c}_ny_n\equiv 0.$$

As the constants $\hat{c}_i$ were not all zero, this proves that $y_1,\dots,y_n$ are linearly dependent on $\mathbb{T}$. $\square$

The solutions of (3.1) turn out to be somewhat complicated. To simplify things, we introduce the following notation. For $1 \le i \le n$, $i \le j \le n$, define the doubly-indexed family of functions $\{y_{i,j}\}$ by

$$y_{i,i}(t) = e_{\frac{\lambda_i}{a_i}}(t, t_0),$$

and for $i+1 \le j \le n$,

$$y_{i,j}(t) = e_{\frac{\lambda_j}{a_j}}(t, t_0) \int_{t_0}^{t} \left(\frac{1}{a_j(s) + \mu(s)\lambda_j} \right) e_{\ominus \frac{\lambda_j}{a_j}}(s, t_0) y_{i,j-1}(s) \, \Delta s.$$

Speaking informally, the first subscript tells you which exponential function to start with, then you proceed using the recursive formula described above. The second subscript tells you when to stop. Note also that the expression $\frac{1}{a_i(t)+\mu(t)\lambda_i}$ may be rewritten as follows:

$$\frac{1}{a_i(t) + \mu(t)\lambda_i} = \begin{cases} -\frac{1}{\lambda_i}\left(\ominus \frac{\lambda_i}{a_i(t)}\right) & \lambda_i \ne 0 \\ \frac{1}{a_i(t)} & \lambda_i = 0. \end{cases}$$

Occasionally, we will make use of this simplification.

Example 71. Suppose $n = 3$. Then

$$y_{3,3} = e_{\frac{\lambda_3}{a_3}}(t, t_0),$$

$$y_{2,3} = e_{\frac{\lambda_3}{a_3}}(t, t_0) \int_{t_0}^{t} \left(\frac{1}{a_3(s) + \mu(s)\lambda_3} \right) e_{\ominus \frac{\lambda_3}{a_3}}(s, t_0) e_{\frac{\lambda_2}{a_2}}(s, t_0) \, \Delta s,$$

and

$$\begin{aligned} y_{1,3} \;=\; & e_{\frac{\lambda_3}{a_3}}(t, t_0) \int_{t_0}^{t} \Bigg[\left(\frac{1}{a_3(s) + \mu(s)\lambda_3} \right) e_{\ominus \frac{\lambda_3}{a_3}}(s, t_0) e_{\frac{\lambda_2}{a_2}}(s, t_0) \\ & \int_{t_0}^{s} \left(\frac{1}{a_2(\tau) + \mu(\tau)\lambda_2} \right) e_{\ominus \frac{\lambda_2}{a_2}}(\tau, t_0) e_{\frac{\lambda_1}{a_1}}(\tau, t_0) \, \Delta\tau \Bigg] \, \Delta s. \end{aligned}$$

Theorem 72. *Suppose* (3.1) *is regressive. Then* $\{y_{i,n}\}_{i=1}^{n}$ *are linearly-independent solutions of* (3.1) *are , and a general solution of* (3.1) *is given by*

$$\sum_{i=1}^{n} \alpha_i y_{i,n},$$

where α_i *are constants,* $1 \le i \le n$.

Proof. As (3.1) is regressive, $e_{\frac{\lambda_i}{a_i}}$ is well defined for $1 \le i \le n$, and thus the functions $\{y_{i,n}\}_{i=1}^{n}$ exist. It is relatively straightforward to demonstrate that these functions

are in $\mathbb{D}$. We will prove the remainder of the statement by induction on n. (Note, the proof is by induction on n, the order of the equation, not by induction on i.)

Base case, $n = 1$: Then our dynamic equation has the form

$$(D_{a_1} - \lambda_1)y = 0.$$

Since we only have one (nontrivial) function, linear independence is clear. It remains to show that $y_{1,1}$ satisfies the dynamic equation. Substitution gives

$$\begin{aligned}
(D_{a_1} - \lambda_1)y_{1,1}(t) &= a_1(t)y_{1,1}^{\Delta}(t) - \lambda_1 y_{1,1}(t) \\
&= a_1(t)e_{\frac{\lambda_1}{a_1}}^{\Delta}(t, t_0) - \lambda_1 e_{\frac{\lambda_1}{a_1}}(t, t_0) \\
&= a_1(t)\frac{\lambda_1}{a_1(t)}e_{\frac{\lambda_1}{a_1}}(t, t_0) - \lambda_1 e_{\frac{\lambda_1}{a_1}}(t, t_0) \\
&= \lambda_1 e_{\frac{\lambda_1}{a_1}}(t, t_0) - \lambda_1 e_{\frac{\lambda_1}{a_1}}(t, t_0) \\
&= 0,
\end{aligned}$$

and we have successfully established the base case.

Induction step: Now, assume that $\{y_{i,n-1}\}_{i=1}^{n-1}$ are linearly independent solutions of

$$(D_{a_1} - \lambda_1)(D_{a_2} - \lambda_2)\ldots(D_{a_{n-1}} - \lambda_{n-1})y = 0,$$

and consider $\{y_{i,n}\}_{i=1}^{n}$. We will first show that these functions are solutions of

$$Ly = (D_{a_1} - \lambda_1)(D_{a_2} - \lambda_2)\ldots(D_{a_n} - \lambda_n)y = 0.$$

Note that

$$Ly = (D_{a_1} - \lambda_1)\ldots(D_{a_{n-1}} - \lambda_{n-1})\left[(D_{a_n} - \lambda_n)y\right].$$

Now, our functions take on different forms depending on whether $i = n$, or $1 \leq i \leq n-1$. Therefore, we must consider these cases separately.

Case 1, $i = n$: Then, ignoring the argument, t,

$$\begin{aligned}
Ly_{n,n} &= (D_{a_1} - \lambda_1)\ldots(D_{a_{n-1}} - \lambda_{n-1})\left[(D_{a_n} - \lambda_n)y_{n,n}\right] \\
&= (D_{a_1} - \lambda_1)\ldots(D_{a_{n-1}} - \lambda_{n-1})\left[a_n y_{n,n}^{\Delta} - \lambda_n y_{n,n}\right] \\
&= (D_{a_1} - \lambda_1)\ldots(D_{a_{n-1}} - \lambda_{n-1})\left[a_n e_{\frac{\lambda_n}{a_n}}^{\Delta} - \lambda_n e_{\frac{\lambda_n}{a_n}}\right] \\
&= (D_{a_1} - \lambda_1)\ldots(D_{a_{n-1}} - \lambda_{n-1})\left[a_n \frac{\lambda_n}{a_n} e_{\frac{\lambda_n}{a_n}} - \lambda_n e_{\frac{\lambda_n}{a_n}}\right] \\
&= (D_{a_1} - \lambda_1)\ldots(D_{a_{n-1}} - \lambda_{n-1})(0) \\
&= 0.
\end{aligned}$$

Case 2, $1 \leq i \leq n-1$: Then $y_{i,n}$ has the form

$$y_{i,n}(t) = e_{\frac{\lambda_n}{a_n}}(t,t_0) \int_{t_0}^{t} \left(\frac{1}{a_n(s) + \mu(s)\lambda_n} \right) e_{\ominus \frac{\lambda_n}{a_n}}(s,t_0) y_{i,n-1}(s)\, \Delta s$$

and hence

$$\begin{aligned}
y_{i,n}^{\Delta}(t) &= \frac{\lambda_n}{a_n(t)} e_{\frac{\lambda_n}{a_n}}(t,t_0) \int_{t_0}^{t} \left(\frac{1}{a_n(s) + \mu(s)\lambda_n} \right) e_{\ominus \frac{\lambda_n}{a_n}}(s,t_0) y_{i,n-1}(s)\, \Delta s \\
&\quad + e^{\sigma}_{\frac{\lambda_n}{a_n}}(t,t_0) \left(\frac{1}{a_n(t) + \mu(t)\lambda_n} \right) e_{\ominus \frac{\lambda_n}{a_n}}(t,t_0) y_{i,n-1}(t) \\
&= \frac{\lambda_n}{a_n(t)} e_{\frac{\lambda_n}{a_n}}(t,t_0) \int_{t_0}^{t} \left(\frac{1}{a_n(s) + \mu(s)\lambda_n} \right) e_{\ominus \frac{\lambda_n}{a_n}}(s,t_0) y_{i,n-1}(s)\, \Delta s \\
&\quad + \left(1 + \mu(t) \frac{\lambda_n}{a_n(t)} \right) e_{\frac{\lambda_n}{a_n}}(t,t_0) \left(\frac{1}{a_n(t) + \mu(t)\lambda_n} \right) e_{\ominus \frac{\lambda_n}{a_n}}(t,t_0) y_{i,n-1}(t) \\
&= \frac{\lambda_n}{a_n(t)} y_{i,n}(t) + \left(\frac{a_n(t) + \mu(t)\lambda_n}{a_n(t)} \right) \left(\frac{1}{a_n(t) + \mu(t)\lambda_n} \right) y_{i,n-1}(t) \\
&= \frac{\lambda_n}{a_n(t)} y_{i,n}(t) + \frac{1}{a_n(t)} y_{i,n-1}(t) \\
&= \frac{1}{a_n(t)} \left(\lambda_n y_{i,n}(t) + y_{i,n-1}(t) \right).
\end{aligned}$$

Therefore, we see that

$$\begin{aligned}
(D_{a_n} - \lambda_n) y_{i,n}(t) &= a_n(t) y_{i,n}^{\Delta}(t) - \lambda_n y_{i,n}(t) \\
&= a_n(t) \frac{1}{a_n(t)} \left(\lambda_n y_{i,n}(t) + y_{i,n-1}(t) \right) - \lambda_n y_{i,n}(t) \\
&= \lambda_n y_{i,n}(t) + y_{i,n-1}(t) - \lambda_n y_{i,n}(t) \\
&= y_{i,n-1}(t),
\end{aligned}$$

and finally, we have

$$\begin{aligned}
L y_{i,n} &= (D_{a_1} - \lambda_1) \ldots (D_{a_{n-1}} - \lambda_{n-1}) \left[(D_{a_n} - \lambda_n) y_{i,n} \right] \\
&= (D_{a_1} - \lambda_1) \ldots (D_{a_{n-1}} - \lambda_{n-1}) y_{i,n-1} \\
&= 0,
\end{aligned}$$

and we see that the dynamic equation (3.1) is satisfied. It remains to show that the functions $\{y_{i,n}\}_{i=1}^{n}$ are linearly independent. To see this, consider the Wronskian,

$W(y_{n,n},\ldots,y_{1,n})$. We have

$$W=\det\begin{bmatrix} y_{n,n} & y_{n-1,n} & \cdots & y_{1,n} \\ (D_{a_n}-\lambda_n)y_{n,n} & (D_{a_n}-\lambda_n)y_{n-1,n} & \cdots & (D_{a_n}-\lambda_n)y_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ (D_{a_2}-\lambda_2)\ldots(D_{a_n}-\lambda_n)y_{n,n} & (D_{a_2}-\lambda_2)\ldots(D_{a_n}-\lambda_n)y_{n-1,n} & \cdots & (D_{a_2}-\lambda_2)\ldots(D_{a_n}-\lambda_n)y_{1,n} \end{bmatrix}.$$

Now, by previous calculations, we know that $(D_{a_n}-\lambda_n)y_{n,n}=0$, and for $1\le i\le n-1$, $(D_{a_n}-\lambda_n)y_{i,n}=y_{i,n-1}$. Using these facts to simplify our matrix, we get

$$W=\det\begin{bmatrix} y_{n,n} & y_{n-1,n} & \cdots & y_{1,n} \\ 0 & y_{n-1,n-1} & \cdots & y_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (D_{a_2}-\lambda_2)\ldots(D_{a_{n-1}}-\lambda_{n-1})y_{n-1,n-1} & \cdots & (D_{a_2}-\lambda_2)\ldots(D_{a_{n-1}}-\lambda_{n-1})y_{1,n-1} \end{bmatrix}.$$

We now expand this determinant about the first column, to get

$$W=y_{n,n}\det\begin{bmatrix} y_{n-1,n-1} & \cdots & y_{1,n-1} \\ \vdots & \ddots & \vdots \\ (D_{a_2}-\lambda_2)\ldots(D_{a_{n-1}}-\lambda_{n-1})y_{n-1,n-1} & \cdots & (D_{a_2}-\lambda_2)\ldots(D_{a_{n-1}}-\lambda_{n-1})y_{1,n-1} \end{bmatrix}.$$

Looking closer, we see that what we really have is

$$W(y_{n,n},\ldots,y_{1,n})=y_{n,n}W(y_{n-1,n-1},\ldots,y_{1,n-1}).$$

Now $W(y_{n-1,n-1},\ldots,y_{1,n-1})$ is never zero, since, by our induction hypothesis, we have assumed these functions are linearly independent. And, as exponential functions are never zero, we have that $y_{n,n}$ is also never zero. Thus we can conclude that $W(y_{n,n},\ldots,y_{1,n})$ is never zero, and hence the functions $\{y_{i,n}\}_{i=1}^{n}$ are linearly independent. By induction, the proof is complete. □

3.1.2 Special Cases

In this section, we will look at how this theory simplifies in the case where all of the functions, a_i, $1\le i\le n$ are the same. We will denote this common function simply by a. Formally, let $a:\mathbb{T}\to\mathbb{R}$ be rd-continuous, and assume that $a(t)\ne 0$ for any $t\in\mathbb{T}$. Define the operator, D_a, by

$$[D_a y](t):=a(t)y^{\Delta}(t).$$

Clearly, D_a is a linear operator. Now consider the dynamic equation

$$L_1 w=0, \quad \text{where} \quad L_1 w=c_n D_a^n w+c_{n-1}D_a^{n-1}w+\cdots+c_1 D_a w+c_0 w, \tag{3.5}$$

and c_i is constant, $1\le i\le n$, $c_n\ne 0$. As equation (3.5) can be multiplied by a nonzero constant without changing the solutions, we will usually take $c_n=1$. Here we take our domain $\mathbb{D}_{L_1}$ to be the set of functions $w:\mathbb{T}\to\mathbb{R}$ such that $D_a^{n-1}w:T^{\kappa^{n-1}}\to\mathbb{R}$ is defined and is delta-differentiable, and we say that a function $w\in\mathbb{D}_{L_1}$ is a solution

of (3.5) provided $L_1 w(t) = 0$ for all $t \in \mathbb{T}^{\kappa^n}$. This is consistent with the domain and definition of solution which we used in the general case.

Definition 73. The *characteristic polynomial* associated with $L_1 w$, which we will denote by p, is given by

$$p(\lambda) = c_n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1 \lambda + c_0.$$

The equation

$$p(\lambda) = 0$$

is called the *characteristic equation.*

Definition 74. Denote the n roots (including multiplicity) of the characteristic equation, $p(\lambda) = 0$, by λ_i, $1 \leq i \leq n$. We say the dynamic equation (3.5) is *regressive* provided $\frac{\lambda_i}{a} \in \mathcal{R}$ for $1 \leq i \leq n$.

This definition agrees with the earlier definition of regressivity of (3.1). Note, then, that equation (3.5), with $c_n = 1$, can be written in factored form:

$$L_1 w = \left[\prod_{i=1}^{n} (D_a - \lambda_i) \right] w = 0. \tag{3.6}$$

It is easy to show that these factors commute with one another.

Although the results presented below are valid for any rd-continuous function, a, without zeros, there are two specific choices of a which merit further comment. First, if we choose $a(t) = 1$, then (3.5) is just a constant coefficient dynamic equation. Second, if $a(t) = t$, then (3.5) is an Euler-Cauchy dynamic equation. These equations have been studied extensively by Bohner and Akin-Bohner in [7, 2], and, although our notation is slightly different, the results in this section should be considered to be a generalization of their work.

One of the drawbacks of our definition of regressivity is that the roots of the characteristic equation must be known before it can be determined whether or not the equation is regressive. In the following lemma, we present an equivalent condition for regressivity which does not require the roots of the characteristic equation to be determined explicitly.

Lemma 75. *Equation* (3.5) *is regressive if and only if*

$$\sum_{i=0}^{n} c_i (a(t))^i (-\mu(t))^{n-i} \neq 0 \quad \textit{for all} \quad t \in \mathbb{T}.$$

The proof essentially mirrors the proof presented by Bohner and Akin-Bohner for Euler equations [7, Theorem 2.15], and we do not include it here.

Theorem 76. *Let m be the number of* distinct *roots of the characteristic equation associated with $L_1 w = 0$. Denote these roots by r_i, $1 \leq i \leq m$. For $1 \leq i \leq m$, let $m(r_i)$ denote the multiplicity of the root, r_i. Then, for $1 \leq i \leq m$, $\{w_{i,j}\}_{j=1}^{m(r_i)}$ defined by*

$$w_{i,j}(t) = e_{\frac{r_i}{a}}(t, t_0) z_{i,j}(t),$$

where

$$z_{i,1}(t) = 1, \quad \text{and} \quad z_{i,j+1} = \int_{t_0}^{t} \frac{1}{a(s) + \mu(s) r_i} z_{i,j}(s) \, \Delta s \quad \text{for} \quad 1 \leq j \leq m(r_i) - 1,$$

are $m(r_i)$ linearly independent solutions of $L_1 w = 0$ associated with r_i.

Moreover, if $\alpha_{i,j}$ are arbitrary constants, then a general solution of $L_1 w = 0$ is given by

$$w(t) = \sum_{i=1}^{m} \sum_{j=1}^{m(r_i)} \alpha_{i,j} w_{i,j}(t).$$

Before embarking on the proof of this theorem, we will introduce some notation that will be needed, and we will establish a lemma, which will also be useful in the proof.

Recall that our dynamic equation can be written in factored form. (See equation (3.6).) Although the factors commute, for the time being, we will fix the order of the factors, and write the equation as

$$(D_a - \lambda_1)(D_a - \lambda_2) \ldots (D_a - \lambda_n) w = 0.$$

Note that although the constants λ_i need not be distinct, we have that for $1 \leq i \leq n$, $\lambda_i = r_j$ for some $1 \leq j \leq m$. Now for $1 \leq i \leq m$, $1 \leq k \leq n$, define the set $\varphi_k(r_i)$ by

$$\varphi_k(r_i) = \{ j : \lambda_j = r_i, 1 \leq j \leq k \},$$

and let

$$m_k(r_i) = |\varphi_k(r_i)|,$$

where $|\cdot|$ denotes the order of the set. In other words, $m_k(r_i)$ is the number of times r_i appears in the first k factors, and $m_n(r_i) = m(r_i)$. Furthermore, for $1 \leq k \leq n$, define

$$W_k = \bigcup_{i=1}^{m} \bigcup_{j=1}^{m_k(r_i)} \{w_{i,j}\}.$$

(If $m_k(r_i) = 0$ for some i, we understand $\bigcup_{j=1}^{0} \{w_{i,j}\}$ to be empty for that value of i.) Note that W_k contains exactly k elements, and that $W_k \subset W_{k+1}$ for $1 \leq k \leq n-1$.

Lemma 77. *For $1 \leq k \leq n-1$, if $w_{i,j} \in W_k$, then*

$$u_{i,j,k}(t) := e_{\frac{\lambda_{k+1}}{a}}(t,t_0) \int_{t_0}^{t} \frac{1}{a(s)+\mu(s)\lambda_{k+1}} e_{\ominus\frac{\lambda_{k+1}}{a}}(s,t_0) w_{i,j}(s)\, \Delta s$$

belongs to the span of W_{k+1}, denoted by $S[W_{k+1}]$.

Proof. Fix k with $1 \leq k \leq n-1$. We must show that $u_{i,j,k}$, as defined in the statement of the theorem, is in $S[W_{k+1}]$ for all $w_{i,j} \in W_k$. That is, we must show $u_{i,j,k} \in S[W_{k+1}]$ whenever $1 \leq i \leq m$, and $1 \leq j \leq m_k(r_i)$. Not that if $m_k(r_i) = 0$ for some i, then the statement is vacuous. Therefore, fix i with $1 \leq i \leq m$, and assume $m_k(r_i) \geq 1$. We now proceed by induction on j.

Base case, $j = 1$: If $j = 1$, then

$$w_{i,j}(t) = w_{i,1}(t) = e_{\frac{r_i}{a}}(t,t_0) z_{i,1}(t) = e_{\frac{r_i}{a}}(t,t_0).$$

So, we get

$$u_{i,j,k}(t) = e_{\frac{\lambda_{k+1}}{a}}(t,t_0) \int_{t_0}^{t} \frac{1}{a(s)+\mu(s)\lambda_{k+1}} e_{\ominus\frac{\lambda_{k+1}}{a}}(s,t_0) e_{\frac{r_i}{a}}(s,t_0)\, \Delta s.$$

Now this expression simplifies differently, depending on whether or not $\lambda_{k+1} = r_i$, so we consider these cases separately.

Case 1, $\lambda_{k+1} = r_i$: In this case, it is clear that $m_{k+1}(r_i) \geq 2$, and we get

$$u_{i,j,k}(t) = e_{\frac{r_i}{a}}(t,t_0) \int_{t_0}^{t} \frac{1}{a(s)+\mu(s) r_i}\, \Delta s = w_{i,2}(t) \in W_{k+1}.$$

Case 2, $\lambda_{k+1} \neq r_i$: Choose q with $1 \leq q \leq m$ such that $\lambda_{k+1} = r_q$. In this case, we see that $m_{k+1}(r_i) \geq 1$, and $m_{k+1}(r_q) \geq 1$. Furthermore,

$$\begin{aligned}
u_{i,j,k}(t) &= e_{\frac{r_q}{a}}(t,t_0) \int_{t_0}^{t} \frac{1}{a(s)} \left(\frac{1}{1+\mu(s)\frac{r_q}{a(s)}} \right) \frac{e_{\frac{r_i}{a}}(s,t_0)}{e_{\frac{r_q}{a}}(s,t_0)}\, \Delta s \\
&= e_{\frac{r_q}{a}}(t,t_0) \int_{t_0}^{t} \frac{1}{a(s)} \frac{e_{\frac{r_i}{a}}(s,t_0)}{e^{\sigma}_{\frac{r_q}{a}}(s,t_0)}\, \Delta s \\
&= \frac{1}{r_i - r_q} e_{\frac{r_q}{a}}(t,t_0) \int_{t_0}^{t} \left[e_{\frac{r_i}{a}\ominus\frac{r_q}{a}}(s,t_0) \right]^{\Delta} \Delta s \\
&= \frac{1}{r_i - r_q} e_{\frac{r_q}{a}}(t,t_0) \left[e_{\frac{r_i}{a}\ominus\frac{r_q}{a}}(s,t_0) \right]_{s=t_0}^{s=t} \\
&= \frac{1}{r_i - r_q} e_{\frac{r_q}{a}}(t,t_0) \left[e_{\frac{r_i}{a}\ominus\frac{r_q}{a}}(t,t_0) - 1 \right] \\
&= \frac{1}{r_i - r_q} \left[e_{\frac{r_i}{a}}(t,t_0) - e_{\frac{r_q}{a}}(t,t_0) \right]
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{r_i - r_q}\left[w_{i,1}(t) - w_{q,1}(t)\right] \\ &\in S[W_{k+1}]. \end{aligned}$$

As either Case 1 or Case 2 must occur, we see that this establishes the base case.

Induction step: Now suppose $u_{i,j,k} \in S[W_{k+1}]$, and assume $w_{i,j+1} \in W_k$. (If $w_{i,j+1} \notin W_k$, there is nothing to show.) Then, we must show $u_{i,j+1,k} \in S[W_{k+1}]$. In this case, we have

$$w_{i,j+1}(t) = e_{\frac{r_i}{a}}(t,t_0)z_{i,j+1}(t).$$

Thus,

$$u_{i,j+1,k}(t) = e_{\frac{\lambda_{k+1}}{a}}(t,t_0)\int_{t_0}^{t} \frac{1}{a(s)+\mu(s)\lambda_{k+1}} e_{\ominus\frac{\lambda_{k+1}}{a}}(s,t_0)e_{\frac{r_i}{a}}(s,t_0)z_{i,j+1}(s)\ \Delta s$$

As before, this expression simplifies differently depending on whether or not $\lambda_{k+1} = r_i$, so we consider these cases separately.

Case 1, $\lambda_{k+1} = r_i$: In this case,

$$u_{i,j+1,k}(t) = e_{\frac{r_i}{a}}(t,t_0)\int_{t_0}^{t} \frac{1}{a(s)+\mu(s)r_i} z_{i,j+1}(s)\ \Delta s = w_{i,j+2}.$$

Now, as $w_{i,j+1} \in W_k$, we have $m_k(r_i) \geq j+1$, and, since $\lambda_{k+1} = r_i$, we see that $m_{k+1}(r_i) \geq j+2$. Hence $w_{i,j+2} \in W_{k+1}$, which gives $u_{i,j+1,k} \in S[W_{k+1}]$.

Case 2, $\lambda_{k+1} \neq r_i$: In this case,

$$\begin{aligned} u_{i,j+1,k}(t) &= e_{\frac{\lambda_{k+1}}{a}}(t,t_0)\int_{t_0}^{t} \frac{1}{a(s)+\mu(s)\lambda_{k+1}} e_{\ominus\frac{\lambda_{k+1}}{a}}(s,t_0)e_{\frac{r_i}{a}}(s,t_0)z_{i,j+1}(s)\ \Delta s \\ &= \frac{1}{r_i - \lambda_{k+1}} e_{\frac{\lambda_{k+1}}{a}}(t,t_0)\int_{t_0}^{t} \left[e_{\frac{r_i}{a}\ominus\frac{\lambda_{k+1}}{a}}(s,t_0)\right]^{\Delta} z_{i,j+1}(s)\ \Delta s \end{aligned}$$

Now, integrating by parts, we get

$$\begin{aligned} u_{i,j+1,k}(t) &= \frac{1}{r_i - \lambda_{k+1}} e_{\frac{\lambda_{k+1}}{a}}(t,t_0)\left[\left(e_{\frac{r_i}{a}\ominus\frac{\lambda_{k+1}}{a}}(s,t_0)z_{i,j+1}(s)\right)\Big|_{s=t_0}^{s=t}\right. \\ &\qquad \left. - \int_{t_0}^{t} e^{\sigma}_{\frac{r_i}{a}\ominus\frac{\lambda_{k+1}}{a}}(s,t_0)z^{\Delta}_{i,j+1}(s)\ \Delta s\right]. \end{aligned}$$

Now, note that

$$z_{i,j+1}(t) = \int_{t_0}^{t} \frac{1}{a(s)+\mu(s)r_i} z_{i,j}(s)\ \Delta s,$$

so

$$z_{i,j+1}^{\Delta}(t) = \frac{1}{a(t)+\mu(t)r_i} z_{i,j}(t).$$

Additionally, we get

$$z_{i,j+1}(t_0) = 0.$$

Substituting the expression for $z_{i,j+1}^{\Delta}$ into our expression for $u_{i,j+1,k}$, and making some other simplifications, we get

$$\begin{aligned}
u_{i,j+1,k}(t) &= \frac{1}{r_i-\lambda_{k+1}} e_{\frac{\lambda_{k+1}}{a}}(t,t_0)\Bigg[e_{\frac{r_i}{a}\ominus\frac{\lambda_{k+1}}{a}}(t,t_0) z_{i,j+1}(t) - z_{i,j+1}(t_0) \\
&\quad - \int_{t_0}^{t} e^{\sigma}_{\frac{r_i}{a}}(s,t_0) e^{\sigma}_{\ominus\frac{\lambda_{k+1}}{a}}(s,t_0) \frac{1}{a(s)+\mu(s)r_i} z_{i,j}(s)\,\Delta s\Bigg] \\
&= \frac{1}{r_i-\lambda_{k+1}} e_{\frac{\lambda_{k+1}}{a}}(t,t_0)\Bigg[e_{\frac{r_i}{a}\ominus\frac{\lambda_{k+1}}{a}}(t,t_0) z_{i,j+1}(t) - 0 \\
&\quad - \int_{t_0}^{t} \left(1+\frac{\mu(s)r_i}{a(s)}\right) e_{\frac{r_i}{a}}(s,t_0) e^{\sigma}_{\ominus\frac{\lambda_{k+1}}{a}}(s,t_0) \frac{1}{a(s)+\mu(s)r_i} z_{i,j}(s)\,\Delta s\Bigg] \\
&= \frac{1}{r_i-\lambda_{k+1}}\Bigg[e_{\frac{r_i}{a}}(t,t_0) z_{i,j+1}(t) \\
&\quad - e_{\frac{\lambda_{k+1}}{a}}(t,t_0)\int_{t_0}^{t} \frac{1}{a(s)} e^{\sigma}_{\ominus\frac{\lambda_{k+1}}{a}}(s,t_0) e_{\frac{r_i}{a}}(s,t_0) z_{i,j}(s)\,\Delta s\Bigg] \\
&= \frac{1}{r_i-\lambda_{k+1}}\Bigg[w_{i,j+1}(t) \\
&\quad - e_{\frac{\lambda_{k+1}}{a}}(t,t_0)\int_{t_0}^{t} \frac{1}{a(s)+\mu(s)\lambda_{k+1}} e_{\ominus\frac{\lambda_{k+1}}{a}}(s,t_0) e_{\frac{r_i}{a}}(s,t_0) z_{i,j}(s)\,\Delta s\Bigg] \\
&= \frac{1}{r_i-\lambda_{k+1}}\Bigg[w_{i,j+1}(t) \\
&\quad - e_{\frac{\lambda_{k+1}}{a}}(t,t_0)\int_{t_0}^{t} \frac{1}{a(s)+\mu(s)\lambda_{k+1}} e_{\ominus\frac{\lambda_{k+1}}{a}}(s,t_0) w_{i,j}(s)\,\Delta s\Bigg] \\
&= \frac{1}{r_i-\lambda_{k+1}}\left[w_{i,j+1}(t) - u_{i,j,k}(t)\right]
\end{aligned}$$

By our induction hypothesis, $u_{i,j,k} \in S[W_{k+1}]$, so we see that $u_{i,j+1,k} \in S[W_{k+1}]$. Thus, by induction, we have shown that for $1 \le j \le m_k(r_i)$, $u_{i,j,k} \in S[W_{k+1}]$. Since i and k were arbitrary, this completes the proof of the lemma. □

Now, armed with this lemma, we are ready to prove Theorem 76.

Proof of Theorem 76. By Theorem 72, we know that the n linearly independent solutions of $L_1 w = 0$ are $\{y_{i,n}\}_{i=1}^{n}$, as defined in the previous section. Let Y denote this

set. That is, put $Y = \{y_{i,n}\}_{i=1}^n$. Then, what we are really trying to show is that

$$S[Y] = S[W_n],$$

Where, as before, $S[V]$ denotes the span of the set V. Note that $S[Y]$ is finite dimensional. Specifically, since the elements of Y are linearly independent on $\mathbb{T}$, $S[Y]$ has dimension n. We also know that W_n contains n elements, and therefore $\dim(S[W_n]) \leq n$. Thus we need only show that $Y \subset S[W_n]$, which will yield $S[Y] = S[W_n]$, and, furthermore, will establish the linear independence of the elements of W_n on $\mathbb{T}$.

Actually, we are going to show more than $Y \subset S[W_n]$. We are going to show that for $1 \leq i \leq n$, $i \leq j \leq n$, $y_{i,j} \in S[W_n]$. Fix i, with $1 \leq i \leq n$. We now proceed by induction on j. We are going to show that for $i \leq j \leq n$, $y_{i,j} \in S[W_j]$. Then, since

$$W_j \subset W_{j+1} \subset \ldots \subset W_n,$$

this will establish the desired result

Base Case, $j = i$: We must show that $y_{i,i} \in S[W_i]$. By definition, $y_{i,i}(t) = e_{\frac{\lambda_i}{a}}(t, t_0)$. But $\lambda_i = r_k$ for some $1 \leq k \leq m$, and, clearly, $m_i(r_k) \geq 1$. Hence

$$y_{i,i}(t) = e_{\frac{\lambda_i}{a}}(t, t_0) = e_{\frac{r_k}{a}}(t, t_0) = w_{k,1}(t) \in W_i,$$

so $y_{i,i} \in S[W_i]$.

Induction step: Assume $y_{i,j} \in S[W_j]$. We must show that $y_{i,j+1} \in S[W_{j+1}]$. We have

$$y_{i,j+1}(t) = e_{\frac{\lambda_{j+1}}{a}}(t, t_0) \int_{t_0}^{t} \frac{1}{a(s) + \mu(s)\lambda_{j+1}} e_{\ominus\frac{\lambda_{j+1}}{a}}(s, t_0) y_{i,j}(s)\, \Delta s.$$

Now, as $y_{i,j} \in S[W_j]$, we can write it as a linear combination of elements of W_j:

$$y_{i,j}(t) = \sum_{k=1}^{m} \sum_{p=1}^{m_j(r_k)} \alpha_{k,p} w_{k,p}(t),$$

where $\alpha_{k,p}$ are constants. Thus

$$y_{i,j+1}(t) = e_{\frac{\lambda_{j+1}}{a}}(t, t_0) \int_{t_0}^{t} \frac{1}{a(s) + \mu(s)\lambda_{j+1}} e_{\ominus\frac{\lambda_{j+1}}{a}}(s, t_0) \sum_{k=1}^{m} \sum_{p=1}^{m_j(r_k)} \alpha_{k,p} w_{k,p}(s)\, \Delta s.$$

These sums are finite, so we may move them outside the integral without any trouble,

yielding

$$\begin{aligned} y_{i,j+1}(t) &= \sum_{k=1}^{m}\sum_{p=1}^{m_j(r_k)} \alpha_{k,p}\, e_{\frac{\lambda_{j+1}}{a}}(t,t_0)\int_{t_0}^{t}\frac{1}{a(s)+\mu(s)\lambda_{j+1}}e_{\ominus\frac{\lambda_{j+1}}{a}}(s,t_0)w_{k,p}(s)\,\Delta s. \\ &= \sum_{k=1}^{m}\sum_{p=1}^{m_j(r_k)} \alpha_{k,p}u_{k,p,j}(t), \end{aligned}$$

where $u_{k,p,j}$ is as defined in Lemma 77. By Lemma 77, $u_{k,p,j} \in S[W_{j+1}]$, and therefore, $y_{i,j+1}$ is a linear combination of elements of $S[W_{j+1}]$. Hence $y_{i,j+1} \in S[W_{j+1}]$. By induction, then, we have shown that for $i \leq j \leq n$, $y_{i,j} \in S[W_j]$, which implies that for $i \leq j \leq n$, $y_{i,j} \in S[W_n]$. Our choice of i was arbitrary, so we have established that for $1 \leq i \leq n$, $i \leq j \leq n$, $y_{i,j} \in S[W_n]$. Hence $Y \subset S[W_n]$, and, by our earlier discussion, the proof is complete. □

3.2 Linear Equations with Nabla-Derivatives

In this section, we examine linear equations containing nabla-derivatives which can be written in factored form. As one would expect, the results are analogous to the results developed above for delta-derivatives, and the proofs are essentially the same. Therefore, we will simply state the theorems here, and not repeat the proofs.

Fix $n \in \mathbb{N}$, and for $1 \leq i \leq n$ let $b_i : \mathbb{T} \to \mathbb{R}$ be ld-continuous. Furthermore, for $1 \leq i \leq n$, assume $b_i(t) \neq 0$ for any $t \in \mathbb{T}$. Define the operators N_{b_i}, $1 \leq i \leq n$ by

$$[N_{b_i}y](t) := b_i(t)y^{\nabla}(t).$$

For $1 \leq i \leq n$, we take the domain of N_{b_i} to be the family of all nabla-differentiable functions. Clearly each N_{b_i} is a linear operator.

Now consider the dynamic equation

$$My = 0 \quad \text{where} \quad My = (N_{b_1} - \lambda_1)(N_{b_2} - \lambda_2)\ldots(N_{b_n} - \lambda_n)y, \tag{3.7}$$

and λ_i, $1 \leq i \leq n$ are (possibly complex) constants. We define the domain of the operator M, which we will denote by $\mathbb{D}_M$, to be the set of all functions, $y : \mathbb{T} \to \mathbb{R}$, such that $(N_{b_2} - \lambda_2)\ldots(N_{b_n} - \lambda_n)y : \mathbb{T}_{\kappa^{n-1}} \to \mathbb{R}$ is defined, and is nabla-differentiable. In the case $n = 1$, we understand this to mean that our domain is just the set of all nabla-differentiable functions. We say that $y \in \mathbb{D}_M$ is a solution of (3.7) provided $My(t) = 0$ for all $t \in \mathbb{T}_{\kappa^n}$.

Definition 78. We say the dynamic equation (3.1) is *regressive* provided $\frac{\lambda_i}{b_i} \in \mathcal{R}_\nu$ for $1 \leq i \leq n$.

Theorem 79. *The dynamic equation (3.7) is equivalent to the system*

$$x^{\nabla} = B(t)x, \tag{3.8}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$B(t) = \begin{bmatrix} \frac{\lambda_n}{b_n(t)} & \frac{1}{b_n(t)} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_{n-1}}{b_{n-1}(t)} & \frac{1}{b_{n-1}(t)} & \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \frac{\lambda_2}{b_2(t)} & \frac{1}{b_2(t)} \\ 0 & \cdots & \cdots & 0 & \frac{\lambda_1}{b_1(t)} \end{bmatrix}.$$

Theorem 80. *If (3.7) is regressive, $t_0 \in \mathbb{T}$, $y_i \in \mathbb{R}$, $1 \leq i \leq n$, then the initial value problem*

$$My = 0,$$

$$y(t_0) = y_1, \quad (N_{b_n} - \lambda_n)y(t_0) = y_2, \quad ,\ldots, \quad (N_{b_2} - \lambda_2)\ldots(N_{b_n} - \lambda_n)y(t_0) = y_n \tag{3.9}$$

has a unique solution.

Definition 81. For $1 \leq i \leq n$, let $y_i \in \mathbb{D}_M$. We define the *Wronskian* associated with $My = 0$, $W_N = W_N(y_1, \ldots, y_n)$ by

$$W_N(y_1, \ldots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ N_{b_n}y_1 & N_{b_n}y_2 & \cdots & N_{b_n}y_n \\ N_{b_{n-1}}(N_{b_n}y_1) & N_{b_{n-1}}(N_{b_n}y_2) & \cdots & N_{b_{n-1}}(N_{b_n}y_n) \\ \vdots & \vdots & & \vdots \\ N_{b_2}(\ldots(N_{b_n}y_1)) & N_{b_2}(\ldots(N_{b_n}y_2)) & \cdots & N_{b_2}(\ldots(N_{b_n}y_n)) \end{bmatrix}.$$

Adding (nonzero) multiples of one row to another row does not change the determinant of a matrix, so we may write the Wronskian in the equivalent form

$$W_N = \det \begin{bmatrix} y_1 & \cdots & y_n \\ (N_{b_n} - \lambda_n)y_1 & \cdots & (N_{b_n} - \lambda_n)y_n \\ \vdots & & \vdots \\ (N_{b_2} - \lambda_2)\ldots(N_{b_n} - \lambda_n)y_1 & \cdots & (N_{b_2} - \lambda_2)\ldots(N_{b_n} - \lambda_n)y_n \end{bmatrix}.$$

Theorem 82. *If $y_1, \ldots, y_n$ are solutions of* (3.7), *then either*

(i) $W_N(y_1, \ldots, y_n) \equiv 0$, *or*

(ii) $W_N(y_1, \ldots, y_n) \neq 0$ *for any* $t \in \mathbb{T}$.

The second case occurs if and only if the functions y_i, $1 \leq i \leq n$ are linearly independent.

Now, introduce the following notation. For $1 \leq i \leq n$, $i \leq j \leq n$, define the doubly-indexed family of functions $\{y_{i,j}\}$ by

$$y_{i,i}(t) = \hat{e}_{\frac{\lambda_i}{b_i}}(t, t_0),$$

and for $i+1 \leq j \leq n$,

$$y_{i,j}(t) = \hat{e}_{\frac{\lambda_j}{b_j}}(t, t_0) \int_{t_0}^{t} \left(\frac{1}{b_j(s) - \nu(s)\lambda_j} \right) \hat{e}_{\ominus_\nu \frac{\lambda_j}{b_j}}(s, t_0) y_{i,j-1}(s) \, \nabla s.$$

Theorem 83. *Suppose* (3.7) *is regressive. Then $\{y_{i,n}\}_{i=1}^{n}$ are linearly-independent solutions of* (3.7), *and a general solution of* (3.7) *is given by*

$$\sum_{i=1}^{n} \alpha_i y_{i,n},$$

where α_i are constants, $1 \leq i \leq n$.

As before, we now look at the special case where all of the functions, b_i, $1 \leq i \leq n$ are the same. We will denote this common function simply by b. Let $b : \mathbb{T} \to \mathbb{R}$ be ld-continuous, and assume that $b(t) \neq 0$ for any $t \in \mathbb{T}$. Define the operator, N_b, by

$$[N_b y](t) := b(t) y^{\nabla}(t).$$

Clearly, N_b is a linear operator. Now consider the dynamic equation

$$M_1 w = 0, \quad \text{where} \quad M_1 w = c_n N_b^n w + c_{n-1} N_b^{n-1} w + \cdots + c_1 N_b w + c_0 w, \tag{3.10}$$

and c_i is constant, $1 \leq i \leq n$. Again we will usually take $c_n = 1$. Here we take our domain $\mathbb{D}_{M_1}$ to be the set of functions $w : \mathbb{T} \to \mathbb{R}$ such that $N_b^{n-1} w : T_{\kappa^{n-1}} \to \mathbb{R}$ is defined and is nabla-differentiable, and we say that a function $w \in \mathbb{D}_{M_1}$ is a solution of (3.10) provided $M_1 w(t) = 0$ for all $t \in \mathbb{T}_{\kappa^n}$.

Definition 84. The *characteristic polynomial* associated with $M_1 w$, which we will denote by q, is given by

$$q(\lambda) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0.$$

The equation

$$q(\lambda) = 0$$

is called the *characteristic equation.*

Definition 85. Denote the n roots (including multiplicity) of the characteristic equation, $q(\lambda) = 0$, by λ_i, $1 \leq i \leq n$. We say the dynamic equation (3.10) is *regressive* provided $\frac{\lambda_i}{b} \in \mathcal{R}_\nu$ for $1 \leq i \leq n$.

Note, then, that equation (3.10), with $c_n = 1$, can be written in factored form:

$$M_1 w = \left[\prod_{i=1}^{n} (N_b - \lambda_i) \right] w = 0. \tag{3.11}$$

It is easy to show that these factors commute with one another.

Lemma 86. *Equation* (3.10) *is regressive if and only if*

$$\sum_{i=0}^{n} c_i (b(t))^i (\nu(t))^{n-i} \neq 0 \quad \textit{for all} \quad t \in \mathbb{T}.$$

Theorem 87. *Let m be the number of* distinct *roots of the characteristic equation associated with $M_1 w = 0$. Denote these roots by r_i. For $1 \leq i \leq m$, let $m(r_i)$ denote the multiplicity of the root, r_i. Then, for $1 \leq i \leq m$, $\{w_{i,j}\}_{j=1}^{m(r_i)}$ defined by*

$$w_{i,j}(t) = \hat{e}_{\frac{r_i}{b}}(t, t_0) z_{i,j}(t),$$

where

$$z_{i,1}(t) = 1, \quad \textit{and} \quad z_{i,j+1} = \int_{t_0}^{t} \frac{1}{b(s) - \nu(s) r_i} z_{i,j}(s) \, \nabla s \quad \textit{for} \quad 1 \leq j \leq m(r_i) - 1.$$

are $m(r_i)$ linearly independent solutions, of $M_1 w = 0$ associated with r_i. Moreover, if $\alpha_{i,j}$ are arbitrary constants, then a general solution of $M_1 w = 0$ is given by

$$w(t) = \sum_{i=1}^{m} \sum_{j=1}^{m(r_i)} \alpha_{i,j} w_{i,j}(t).$$

3.3 Equations with Both Delta and Nabla Derivatives

Here, we turn our attention to linear equations containing both delta and nabla derivatives. Things become quite a bit more complicated when these two types of derivatives interact, so here we will only look at second-order dynamic equations. Let

$a_1 : \mathbb{T} \to \mathbb{R}$ be continuous, let $a_2 : \mathbb{T} \to \mathbb{R}$ be rd-continuous, and assume that a_1 and a_2 do not vanish. Now consider the dynamic equation

$$(N_{a_1} - \lambda_1)(D_{a_2} - \lambda_2)y = 0. \tag{3.12}$$

Definition 88. We say (3.12) is *regressive* provided $\frac{\lambda_1}{a_1}$ is ν-regressive and $\frac{\lambda_2}{a_2}$ is regressive.

So, suppose (3.12) is regressive. Now, if we look at the left-most factor, we notice that this factor annihilates the function $\hat{e}_{\frac{\lambda_1}{a_1}}(t, t_0)$. As a_1 is continuous, we may apply Theorem 31 to see that

$$\hat{e}_{\frac{\lambda_1}{a_1}}(t, t_0) = e_{\frac{\lambda_1}{a_1^\sigma - \lambda_1 \mu}}(t, t_0),$$

and this second function is annihilated by $(D_{a_1^\sigma - \lambda_1\mu} - \lambda_1)$. We then theorize that if we replace the factor $(N_{a_1} - \lambda_1)$ by the factor $(D_{a_1^\sigma - \lambda_1\mu} - \lambda_1)$, perhaps we will get a related equation which contains only delta derivatives. This new equation can then be solved using techniques from previous sections. We formalize this result in the following theorem.

Theorem 89. *Suppose* (3.12) *is regressive, and let* $b_1(t) := a_1^\sigma(t) - \lambda_1\mu(t)$. *If* y *is a solution of the dynamic equation*

$$(D_{b_1} - \lambda_1)(D_{a_2} - \lambda_2)y = 0, \tag{3.13}$$

then y *is also a solution of* (3.12).

Proof. Assume (3.12) is regressive, and let $b_1(t) := a_1^\sigma(t) - \lambda_1\mu(t)$. Furthermore, let y be a solution of (3.13). Then by Theorem 72, y has the form

$$y(t) = c_1 e_{\frac{\lambda_2}{a_2}}(t, t_0) + c_2 e_{\frac{\lambda_2}{a_2}}(t, t_0) \int_{t_0}^{t} \ominus \left(\frac{\lambda_2}{a_2(s)} \right) e_{\ominus \frac{\lambda_2}{a_2}}(s, t_0) e_{\frac{\lambda_1}{b_1}}(s, t_0)\, \Delta s.$$

Substituting this expression into (3.12), we see that

$$\begin{aligned} (N_{a_1} - \lambda_1)(D_{a_2} - \lambda_2)y(t) &= (N_{a_1} - \lambda_1) e_{\frac{\lambda_1}{b_1}}(t, t_0) \\ &= (N_{a_1} - \lambda_1) \hat{e}_{\frac{\lambda_1}{a_1}}(t, t_0) \\ &= 0, \end{aligned}$$

and the proof is complete. □

Note that we are *not* asserting that these equations are equivalent, nor do we make any uniqueness claims regarding the solutions of initial value problems associated with (3.12). These issues will require further exploration at a later date.

Of course, there is no particular reason the factor with the nabla derivative was written first. We could just as easily consider a dynamic equation of the form

$$(D_{b_1} - \lambda_1)(N_{b_2} - \lambda_2)y = 0. \tag{3.14}$$

In this case, we assume b_1 is continuous, b_2 is ld-continuous and neither b_1 nor b_2 vanish, and we say that (3.14) is regressive if $\frac{\lambda_1}{b_1}$ is regressive and $\frac{\lambda_2}{b_2}$ is ν-regressive. Again, we will replace the left-most factor by a related factor containing a nabla derivative, and obtain the following result.

Theorem 90. *Suppose* (3.14) *is regressive, and let* $a_1(t) := b_1^{\rho}(t) + \lambda_1\nu(t)$. *If* y *is a solution of the dynamic equation*

$$(N_{a_1} - \lambda_1)(N_{b_2} - \lambda_2)y = 0, \tag{3.15}$$

then y *is also a solution of* (3.14).

The proof is analogous to that of the previous theorem.

Clearly, the results contained in this section are only a beginning. We plan to continue studying the mixed derivative case, and hope to include broader results in future publications.

3.4 More Equations with Mixed Derivatives

In final section of this chapter, we turn our attention to an equation which cannot itself be written in factored form, but which is equivalent to one which can be written in factored form. The second-order dynamic equation

$$Lx = 0, \quad \text{where} \quad Lx = x^{\Delta\Delta} + \alpha x^{\Delta} + \beta x, \quad \text{and} \quad \alpha,\ \beta \in \mathbb{R}, \tag{3.16}$$

has been studied extensively. (See, for example, [6, Section 3.2, 3.3].) Equation (3.16) is said to be regressive if $1 - \alpha\mu(t) + \beta\mu^2(t) \neq 0$ for $t \in \mathbb{T}$, or in other words, if $\beta\mu - \alpha$ is regressive.

In this section, we look at second-order linear dynamic equations of the form

$$Mx = 0, \quad \text{where} \quad Mx = x^{\Delta\nabla} + \alpha x^{\nabla} + \beta x^{\rho} \quad \text{and} \quad \alpha,\ \beta \in \mathbb{R}. \tag{3.17}$$

We take the domain of this operator to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that x is Δ-differentiable on $\mathbb{T}^{\kappa}$, x^{Δ} is ∇-differentiable on $\mathbb{T}^{\kappa}_{\kappa}$, and $x^{\Delta\nabla}$ is ld-continuous on $\mathbb{T}^{\kappa}_{\kappa}$. We say a function $x : \mathbb{T} \to \mathbb{R}$ is a solution of (3.17) if x is in the domain of M, and $Mx(t) = 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$. Equation (3.17) is said to be regressive if $1 - \alpha\nu(t) + \beta\nu^2(t) \neq 0$ for $t \in \mathbb{T}$, or in other words, if $\alpha - \beta\nu$ is ν-regressive. It was demonstrated in Chapter 2 that equation (3.17) can be written in self-adjoint form, and hence solutions of initial value problems for this equation are unique.

Lemma 91. *The dynamic equation* (3.16) *is regressive if and only if the dynamic equation* (3.17) *is regressive.*

Proof. First suppose that (3.16) is regressive. We then wish to show that (3.17) is regressive. That is, we want to show that $\alpha - \beta\nu$ is ν-regressive, or that

$$1 - \nu(t)(\alpha - \beta\nu(t)) \neq 0 \quad \text{for} \quad t \in \mathbb{T}$$

Case 1: t is left-dense. Then $\nu(t) = 0$, and we have

$$1 - \nu(t)(\alpha - \beta\nu(t)) = 1 \neq 0.$$

Case 2: t is left-scattered. Then $\sigma(\rho(t)) = t$. As (3.16) is regressive, we have that $\beta\mu - \alpha$ is regressive. Thus by Lemma 30, $-(\beta\mu - \alpha)^{\rho}$ is ν-regressive, and hence

$$1 - \nu(t)(\alpha - \beta\mu(\rho(t))) \neq 0.$$

Then we have

$$\begin{aligned} 1 - \nu(t)(\alpha - \beta\nu(t)) &= 1 - \nu(t)(\alpha - \beta(t - \rho(t))) \\ &= 1 - \nu(t)(\alpha - \beta(\sigma(\rho(t)) - \rho(t))) \\ &= 1 - \nu(t)(\alpha - \beta\mu(\rho(t))) \\ &\neq 0. \end{aligned}$$

Hence $\alpha - \beta\nu$ is ν-regressive, and thus (3.17) is regressive.

Conversely, suppose that (3.17) is regressive. This time, we seek to show that $\beta\mu - \alpha$ is regressive, or that

$$1 + \mu(t)(\beta\mu(t) - \alpha) \neq 0 \quad \text{for} \quad t \in \mathbb{T}.$$

Case 1: t is right-dense. Then $\mu(t) = 0$, and thus

$$1 + \mu(t)(\beta\mu(t) - \alpha) = 1 \neq 0.$$

Case 2: t is right-scattered. Then $\rho(\sigma(t)) = t$. Since (3.17) is regressive, we have that $\alpha - \beta\nu$ is ν-regressive, and thus by Lemma 30, $-(\alpha - \beta\nu)^{\sigma}$ is regressive. This give us that

$$1 + \mu(t)(\beta\nu(\sigma(t)) - \alpha) \neq 0.$$

Then

$$\begin{aligned} 1 + \mu(t)(\beta\mu(t) - \alpha) &= 1 + \mu(t)(\beta(\sigma(t) - t) - \alpha) \\ &= 1 + \mu(t)(\beta(\sigma(t) - \rho(\sigma(t))) - \alpha) \\ &= 1 + \mu(t)(\beta\nu(\sigma(t)) - \alpha) \\ &\neq 0. \end{aligned}$$

Hence $\beta\mu - \alpha$ is regressive, and thus (3.16) is regressive. □

Theorem 92. *If* (3.16) *is regressive and x is a solution of* (3.16), *then* (3.17) *is regressive and x is a solution of* (3.17).

Proof. Assume (3.16) is regressive and x is a solution of (3.16). The fact that (3.17) is regressive follows immediately from Lemma 91. It remains to show that x is a solution of (3.17).

Now, as x is a solution of $Lx = 0$, we know that x^{Δ} is Δ-differentiable, and hence continuous. Furthermore, since x is a solution of $Lx = 0$, we have that

$$x^{\Delta\Delta} = -\alpha x^{\Delta} - \beta x.$$

In other words, $x^{\Delta\Delta}$ is a linear combination of continuous functions and is therefore continuous. Thus, applying Corollary 28 we see that

$$\begin{aligned} Mx &= x^{\Delta\nabla} + \alpha x^{\nabla} + \beta x^{\rho} \\ &= x^{\Delta\Delta\rho} + \alpha x^{\Delta\rho} + \beta x^{\rho} \\ &= (x^{\Delta\Delta} + \alpha x^{\Delta} + \beta x)^{\rho} \\ &= (Lx)^{\rho} \\ &= 0. \end{aligned}$$

□

Corollary 93. *The solution of* (3.17) *is determined by the roots of the characteristic equation as follows:*

(i) If $\alpha^2 - 4\beta > 0$, then the characteristic equation has distinct real roots, λ_1 and λ_2, and a general solution of (3.17) *is given by*

$$x(t) = c_1 e_{\lambda_1}(t, t_0) + c_2 e_{\lambda_2}(t, t_0).$$

(ii) If $\alpha^2 - 4\beta < 0$, then the characteristic equation has complex conjugate roots $p \pm iq$, and a general solution of (3.17) *is given by*

$$x(t) = c_1 e_p(t, t_0) \cos_{q/1+\mu p}(t, t_0) + c_2 e_p(t, t_0) \sin_{q/1+\mu p}(t, t_0).$$

(iii) If $\alpha^2 - 4\beta = 0$, then the characteristic equation has a repeated real root, λ, and a general solution of (3.17) *is given by*

$$x(t) = c_1 e_{\lambda}(t, t_0) + c_2 e_{\lambda}(t, t_0) \int_{t_0}^{t} \frac{1}{1 + \mu(s)\lambda} \Delta s.$$

Corollary 94. *The regressive equations* (3.16) *and* (3.17) *are equivalent.*

Proof. As demonstrated in Chapter 2, equation (3.17) can be written in self-adjoint form, and therefore, solutions of IVPs for (3.17) are unique. Therefore, although Theorem 92 only establishes one direction of this equivalence, the converse follows from uniqueness of solutions. □

Chapter 4

A Self-Adjoint Matrix Equation on a Time Scale

In this chapter, we are concerned with the self-adjoint matrix equation $[P(t)X^{\Delta}]^{\nabla} + Q(t)X = 0$. This is a generalization of the scalar equation studied in Chapter 2, and many of the results we obtain are analogous to results obtained in the scalar case. The added complexity of the matrix case, however, does provide a couple of surprises along the way.

4.1 Preliminary Results

As in the scalar case, contained in Chapter 2, we begin by establishing some results regarding the interaction of the Δ and ∇ derivatives. We first note that differentiation and integration of matrices is defined in terms of differentiation and integration of the matrix entries, and therefore Theorem 26 and its corollaries carry through in the matrix case without further effort.

The definition of regressivity (or ν-regressivity), however, changes slightly in the matrix case, so extending Lemma 30 and Theorem 31 to the matrix case requires some justification.

Lemma 95. *Let P be an $n \times n$ matrix-valued function on $\mathbb{T}$. Then P is regressive if and only if $-P^{\rho}$ is ν-regressive. Similarly, if Q is an $n \times n$ matrix-valued function on $\mathbb{T}$, then Q is ν-regressive if and only if $-Q^{\sigma}$ is regressive.*

Proof. We will only prove the first statement. The proof of the second statement is similar.

First, assume the $n \times n$ matrix-valued function, P, is regressive. We then wish to show that $I + \nu(t)(P^{\rho}(t))$ is invertible.

Case 1: Fix $t \in \mathbb{T}_A$. Then $\rho(t) \in \mathbb{T}$, and as P is regressive, we have that

$$I + \mu(\rho(t))P(\rho(t))$$

is invertible. Using the definition of $\mu(t)$,

$$I + [\sigma(\rho(t)) - \rho(t)]P(\rho(t))$$

is invertible. But $t \in \mathbb{T}_A$, so $\sigma(\rho(t)) = t$, and we get

$$I + [t - \rho(t)]P^\rho(t) = 1 + \nu(t)P^\rho(t)$$

is invertible, as desired.
Case 2: Fix $t \in A$. Then t is left-dense and right-scattered, so $\nu(t) = 0$. Hence

$$I + \nu(t)P^\rho(t) = I + 0P^\rho(t) = I,$$

which is invertible. As $I + \nu(t)P^\rho(t)$ is invertible for any $t \in \mathbb{T}$, we see that $-P^\rho$ is ν-regressive.

Conversely, suppose $-P^\rho$ is ν-regressive. We then wish to show that $I + \mu(t)P(t)$ is invertible.
Case 1: Fix $t \in \mathbb{T}_B$. Then $\sigma(t) \in \mathbb{T}$, and, as $-P^\rho$ is ν-regressive, we have that

$$I + \nu(\sigma(t))P^\rho(\sigma(t))$$

is invertible. Using the definition of $\nu(t)$,

$$I + [\sigma(t) - \rho(\sigma(t))]P(\rho(\sigma(t)))$$

is invertible. But $t \in \mathbb{T}_B$, so $\rho(\sigma(t)) = t$, and we get

$$I + [\sigma(t) - t]P(t) = I + \mu(t)P(t)$$

is invertible, as desired.

Case 2: Fix $t \in B$. Then t is right-dense and left-scattered, so $\mu(t) = 0$. Hence

$$I + \mu(t)P(t) = I + 0P(t) = I,$$

which is invertible. As $1 + \mu(t)P(t)$ is invertible for any $t \in \mathbb{T}$, we see that P is regressive. $\square$

Theorem 96 (Equivalence of delta and nabla exponential functions). *If the $n \times n$ matrix-valued function P is continuous and regressive, then*

$$e_P(t, t_0) = \hat{e}_{\ominus_\nu(-P^\rho)}(t, t_0).$$

If the $n \times n$ matrix-valued function Q is continuous and ν-regressive, then

$$\hat{e}_Q(t, t_0) = e_{\ominus(-Q^\sigma)}(t, t_0).$$

Proof. We will only prove the first statement. The proof of the second statement is

similar. Suppose that the $n \times n$ matrix-valued function P is continuous and regressive, then by Lemma 30 we have that $-P^\rho$ is ν-regressive. Furthermore, since P is continuous, $-P^\rho$ is ld-continuous. Hence $-P^\rho \in \mathcal{R}_\nu$. Then as $\mathcal{R}_\nu$ is an Abelian group under $\oplus_\nu$, we see that $\ominus_\nu(-P^\rho) \in \mathcal{R}_\nu$, and therefore $\hat{e}_{\ominus_\nu(-P^\rho)}(t, t_0)$ exists.

To complete the proof, then, it suffices to show that $e_P(t, t_0)$ solves the initial value problem

$$Y^\nabla = \ominus_\nu(-P^\rho)Y, \quad Y(t_0) = I.$$

Note first that

$$e_P(t_0, t_0) = I.$$

Furthermore, $e_P^\Delta(t, t_0) = P(t)e_P(t, t_0)$, which is continuous. Hence by Corollary 28, $e_P^\nabla(t, t_0) = e_P^{\Delta\rho}(t, t_0)$, and we get

$$\begin{aligned} e_P^\nabla(t, t_0) &= e_P^{\Delta\rho}(t, t_0) \\ &= P^\rho(t)e_P^\rho(t, t_0) \\ &= P^\rho(t)\left[e_P(t, t_0) - \nu(t)e_P^\nabla(t, t_0)\right]. \end{aligned}$$

Rearranging this equation gives

$$[I + \nu(t)P^\rho(t)]\, e_P^\nabla(t, t_0) = P^\rho(t)e_p(t, t_0),$$

so

$$\begin{aligned} e_P^\nabla(t, t_0) &= [I + \nu(t)P^\rho(t)]^{-1}P^\rho(t)e_P(t, t_0) \\ &= \ominus_\nu(-P^\rho)e_P(t, t_0), \end{aligned}$$

and the proof is complete. □

4.2 Abel's Formula and Reduction of Order

Let P and Q be Hermitian $n \times n$ matrix-valued functions on a time scale $\mathbb{T}$. Furthermore, assume that P is continuous, Q is ld-continuous, and $P(t)$ is invertible for all $t \in \mathbb{T}$. Here we are concerned with the second-order matrix equation

$$LX = 0, \quad \text{where} \quad LX = [P(t)X^\Delta]^\nabla + Q(t)X. \tag{4.1}$$

We take the domain of L, denoted by $\mathbb{D}$, to be the set of all $n \times n$ matrix-valued functions X defined on $\mathbb{T}$ such that X is delta-differentiable on $\mathbb{T}^\kappa$, X^Δ is continuous, PX^Δ is nabla-differentiable on $\mathbb{T}^\kappa_\kappa$, and $[PX^\Delta]^\nabla$ is ld-continuous. We say X is a solution of the matrix equation (4.1) on $\mathbb{T}$ provided $X \in \mathbb{D}$ and $LX(t) = 0$ for all $t \in \mathbb{T}^\kappa_\kappa$.

Next, put

$$Z = \begin{bmatrix} X \\ PX^{\Delta} \end{bmatrix} \text{ on } \mathbb{T}^{\kappa} \quad \text{and} \quad S = \begin{bmatrix} \nu(P^{\rho})^{-1}Q & (P^{\rho})^{-1} \\ -Q & 0 \end{bmatrix} \text{ on } \mathbb{T}.$$

Lemma 97. *The matrix-valued function X solves* (4.1) *if and only if Z is nabla-differentiable and solves*

$$Z^{\nabla} = S(t)Z.$$

Proof. Suppose X solves (4.1). We first show that Z is nabla-differentiable. Since X is a solution of (4.1), we have $X \in \mathbb{D}$, and therefore PX^{Δ} is nabla-differentiable. Furthermore, X^{Δ} is continuous, so by Corollary 28, we have X is nabla-differentiable. Hence Z is nabla-differentiable.

It remains to show $Z^{\nabla} = S(t)Z$. We again apply Corollary 28 which gives $X^{\nabla} = X^{\Delta\rho}$. Then

$$P^{\rho}X^{\nabla} = P^{\rho}X^{\Delta\rho} = (PX^{\Delta})^{\rho} = PX^{\Delta} - \nu(PX^{\Delta})^{\nabla}.$$

Additionally, $LX = 0$, and thus we have $(PX^{\Delta})^{\nabla} = -QX$. Substituting into the above expression, we see that

$$P^{\rho}X^{\nabla} = PX^{\Delta} + \nu QX.$$

Multiplying on the left by $(P^{\rho})^{-1}$ then gives

$$X^{\nabla} = (P^{\rho})^{-1}(PX^{\Delta}) + \nu(P^{\rho})^{-1}QX.$$

So

$$\begin{aligned} Z^{\nabla} &= \begin{bmatrix} X^{\nabla} \\ (PX^{\Delta})^{\nabla} \end{bmatrix} \\ &= \begin{bmatrix} (P^{\rho})^{-1}(PX^{\Delta}) + \nu(P^{\rho})^{-1}QX \\ -QX \end{bmatrix} \\ &= \begin{bmatrix} \nu(P^{\rho})^{-1}Q & (P^{\rho})^{-1} \\ -Q & 0 \end{bmatrix} \begin{bmatrix} X \\ PX^{\Delta} \end{bmatrix} \\ &= SZ. \end{aligned}$$

Conversely, suppose Z is nabla-differentiable and solves

$$Z^{\nabla} = S(t)Z.$$

Let Z_1 and Z_2 denote the components of Z. We claim $X = Z_1$ is a solution of (4.1). We first show that $Z_1 \in \mathbb{D}$. As Z is nabla-differentiable, both Z_1 and Z_2 are nabla-differentiable, hence continuous. Furthermore, since $Z^{\nabla} = S(t)Z$, we have

$$Z_1^{\nabla} = \nu(P^{\rho})^{-1}QZ_1 + (P^{\rho})^{-1}Z_2$$

$$\begin{aligned} &= \nu(P^\rho)^{-1}(-Z_2^\nabla) + (P^\rho)^{-1}Z_2 \\ &= (P^\rho)^{-1}(Z_2 - \nu Z_2^\nabla) \\ &= (P^\rho)^{-1}(Z_2^\rho), \end{aligned}$$

which is ld-continuous. Applying Theorem 26, we see that Z_1 is delta-differentiable, and

$$Z_1^\Delta(t) = \begin{cases} Z_1^\nabla(\sigma(t)) & t \in \mathbb{T}_B \\ \lim_{s\to t^+} Z_1^\nabla(s) & t \in B. \end{cases}$$

To simplify this, first consider $t \in \mathbb{T}_B$. Recall that B denoted the set of points in $\mathbb{T}$ which were both right-dense and left-scattered. Hence for $t \in \mathbb{T}_B$, we have $\rho(\sigma(t)) = t$, and therefore, at these points,

$$Z_1^\Delta(t) = Z_1^\nabla(\sigma(t)) = ((P^\rho)^{-1}Z_2^\rho)(\sigma(t)) = (P^{-1}Z_2)(t).$$

Now, if $t \in B$, then t is right-dense and left-scattered, and we get

$$Z_1^\Delta(t) = \lim_{s\to t^+} Z_1^\nabla(s) = \lim_{s\to t^+} ((P^\rho)^{-1}Z_2^\rho)(s) = (P^{-1}Z_2)(t).$$

Since we got the same expression in both cases, we conclude that

$$Z_1^\Delta = P^{-1}Z_2.$$

Since both P and Z_2 are continuous, Z_1^Δ is continuous. Furthermore, $PZ_1^\Delta = Z_2$ is nabla-differentiable, and $[PZ_1^\Delta]^\nabla = Z_2^\nabla = -QZ_1$, which is ld-continuous. Hence $Z_1 \in \mathbb{D}$.

Finally, consider LZ_1. We get

$$\begin{aligned} LZ_1 &= [PZ_1^\Delta]^\nabla + QZ_1 \\ &= Z_2^\nabla + QZ_1 \\ &= -QZ_1 + QZ_1 \\ &= 0, \end{aligned}$$

which completes the proof. □

We have already pointed out that S is ld-continuous. In addition, it is also ν-regressive. To see this, note that

$$[I - \nu S]^{-1} = \begin{bmatrix} I - \nu^2(P^\rho)^{-1}Q & -\nu(P^\rho)^{-1} \\ \nu Q & I \end{bmatrix}^{-1} = \begin{bmatrix} I & \nu(P^\rho)^{-1} \\ -\nu Q & I - \nu^2 Q(P^\rho)^{-1} \end{bmatrix}.$$

Hence $[I - \nu S]$ is invertible, and therefore S is ν-regressive.

In [3] and [7, Section 3.9] it was shown that under these conditions, the initial value problem

$$Z^\nabla = S(t)Z, \quad Z(t_0) = Z_0$$

has a unique solution. By the equivalence developed above, then, we have shown that if X_0, X_0^Δ are any $n \times n$ matrices, then the initial value problem

$$LX = 0, \quad X(t_0) = X_0, \quad X^\Delta(t_0) = X_0^\Delta$$

has a unique solution.

Definition 98. The unique solution of the IVP

$$LX = 0, \quad X(t_0) = 0, \quad X^\Delta(t_0) = P^{-1}(t_0)$$

is called the *principal solution* of (4.1) at t_0. The unique solution of the IVP

$$LX = 0, \quad X(t_0) = -I, \quad X^\Delta(t_0) = 0$$

is called the *associated solution* of (4.1) at t_0.

Definition 99. For X, $Y \in \mathbb{D}$, we define the *Wronskian matrix* of X and Y by

$$W(X,Y)(t) := X^*(t)P(t)Y^\Delta(t) - [P(t)X^\Delta(t)]^*Y(t),$$

for $t \in \mathbb{T}^\kappa$.

We now establish the Lagrange Identity.

Theorem 100 (Lagrange Identity). *If X, $Y \in \mathbb{D}$, then*

$$X^*(t)LY(t) - [LX(t)]^*Y(t) = [W(X,Y)]^\nabla(t)$$

for $t \in \mathbb{T}^\kappa_\kappa$.

Proof. Let X, $Y \in \mathbb{D}$. Then

$$\begin{aligned}
[W(X,Y)]^\nabla &= \left(X^*PY^\Delta - [PX^\Delta]^*Y\right)^\nabla \\
&= X^*[PY^\Delta]^\nabla + (X^*)^\nabla(PY^\Delta)^\rho - \left([PX^\Delta]^*\right)^\nabla Y - \left([PX^\Delta]^*\right)^\rho Y^\nabla \\
&= X^*[PY^\Delta]^\nabla + (X^\nabla)^*P^\rho Y^{\Delta\rho} - \left([PX^\Delta]^\nabla\right)^* Y - \left((X^\Delta)^*P\right)^\rho Y^\nabla \\
&= X^*[PY^\Delta]^\nabla + (X^\nabla)^*P^\rho Y^\nabla - \left([PX^\Delta]^\nabla\right)^* Y - (X^\nabla)^*P^\rho Y^\nabla \\
&= X^*[PY^\Delta]^\nabla - \left([PX^\Delta]^\nabla\right)^* Y \\
&= X^*[PY^\Delta]^\nabla + X^*QY - \left([PX^\Delta]^\nabla\right)^* Y - X^*QY \\
&= X^*\left([PY^\Delta]^\nabla + QY\right) - \left([PX^\Delta]^\nabla + QX\right)^* Y \\
&= X^*LY - [LX]^*Y
\end{aligned}$$

on $\mathbb{T}^\kappa_\kappa$. ∎

Corollary 101 (Abel's Formula). *If X, Y are solutions of* (4.1) *on* $\mathbb{T}$, *then*

$$W(X,Y)(t) \equiv C,$$

for $t \in \mathbb{T}^\kappa$, where C is a constant matrix.

Proof. Suppose that X and Y are solutions of (4.1) on $\mathbb{T}$. Then by the Lagrange Identity,

$$W^\nabla(X,Y)(t) = 0$$

for all $t \in \mathbb{T}^\kappa_\kappa$. Hence $W(X,Y)$ must be a constant matrix on $\mathbb{T}^\kappa$. □

Theorem 102 (Converse of Abel's Formula). *Suppose U is a solution of* (4.1) *such that $U(t)$ is invertible for all $t \in \mathbb{T}$. If $V \in \mathbb{D}$ satisfies*

$$W(U,V)(t) = C$$

on $\mathbb{T}^\kappa$, where C is a constant matrix, then V is also a solution of (4.1).

Proof. Suppose U is a solution of (4.1) such that $U(t)$ is invertible for all $t \in \mathbb{T}$, and assume that $V \in \mathbb{D}$ satisfies $W(U,V)(t) = C$ for all $t \in \mathbb{T}^\kappa$. Then by Theorem 100, we have

$$U^*(t)LV(t) - [LU(t)]^*V(t) = [W(U,V)]^\nabla(t) = C^\nabla = 0.$$

But U is a solution of (4.1), so $LU = 0$. This gives

$$U^*(t)LV(t) = 0.$$

Multiplication on the left by $(U^{-1})^*$ then gives the desired result. □

Abel's formula shows that if X is a solution of (4.1), then $W(X,X)(t) \equiv C$ for $t \in \mathbb{T}^\kappa$. This leads to the following definition.

Definition 103. (i) If X is a solution of (4.1) such that

$$W(X,X)(t) \equiv 0 \quad \text{for} \quad t \in \mathbb{T}^\kappa,$$

then we say X is a *prepared solution* (or *conjoined solution*) of (4.1).

(ii) If X and Y are two conjoined solutions such that

$$W(X,Y)(t) \equiv I \quad \text{for} \quad t \in \mathbb{T}^\kappa,$$

then we say that X and Y are *normalized conjoined bases* of (4.1).

Theorem 104. *Suppose X is a solution of* (4.1) *on* $\mathbb{T}$. *Then the following are equivalent:*

(i) X is a prepared solution;

(ii) $X^*(t)P(t)X^\Delta(t)$ *is Hermitian for all* $t \in \mathbb{T}^\kappa$;

(iii) $X^*(t_0)P(t_0)X^\Delta(t_0)$ *is Hermitian for some* $t_0 \in \mathbb{T}^\kappa$.

Proof. First, we will show that *(i)* $\implies$ *(ii)*. Assume X is a prepared solution of (4.1) on $\mathbb{T}$. Then

$$W(X,X)(t) = X^*(t)P(t)X^\Delta(t) - [P(t)X^\Delta(t)]^*X(t) = 0,$$

for $t \in \mathbb{T}^\kappa$. This gives

$$X^*(t)P(t)X^\Delta(t) = (X^\Delta)^*(t)P(t)X(t),$$

for $t \in \mathbb{T}^\kappa$, and therefore $X^*(t)P(t)X^\Delta(t)$ is Hermitian for all $t \in \mathbb{T}^\kappa$.

Next, note that *(ii)* $\implies$ *(iii)* is trivial. Therefore, to complete the proof, we need only establish *(iii)* $\implies$ *(i)*. Suppose X is a solution of (4.1) on $\mathbb{T}$, and assume there is some $t_0 \in \mathbb{T}^\kappa$ such that $X^*(t_0)P(t_0)X^\Delta(t_0)$ is Hermitian. Then we have

$$W(X,X)(t_0) = X^*(t_0)P(t_0)X^\Delta(t_0) - [P(t_0)X^\Delta(t_0)]^*X(t_0) = 0.$$

Since Abel's formula tells us that $W(X,X)(t)$ is constant, we then have $W(X,X)(t) = 0$ for all $t \in \mathbb{T}^\kappa$, so X is a prepared solution. □

Remark 105. In the previous theorem, we assumed that X was a prepared solution of (4.1). This assumption was necessary in order to apply Abel's formula and establish the implication *(iii)* $\implies$ *(i)*. To establish *(i)* $\implies$ *(ii)*, however, we did not actually need to know that X was a solution of (4.1), we only needed $X \in \mathbb{D}$ satisfies $W(X,X)(t) \equiv 0$ on $\mathbb{T}$. Therefore, in the following corollary, we include only this weaker assumption, and do not require $LX = 0$ to hold.

Corollary 106. *If* $X \in \mathbb{D}$ *satisfies* $W(X,X)(t) \equiv 0$, *then the following matrices are Hermitian on* $\mathbb{T}^\kappa$.

(i) X^*PX^Δ,

(ii) $(X^\rho)^*P^\rho X^\nabla$,

(iii) $(X^\sigma)^*PX$, *and*

(iv) $X^*P^\rho X^\rho$.

If, in addition, $X(t)$ *is invertible for all* $t \in \mathbb{T}^\kappa$, *then the following matrices are Hermitian on* $\mathbb{T}^\kappa$.

(i) $PX^\sigma X^{-1}$,

(ii) $PX(X^\sigma)^{-1}$,

(iii) $PX^{\Delta}X^{-1}$,

(iv) $P^{\rho}X(X^{\rho})^{-1}$,

(v) $P^{\rho}X^{\rho}X^{-1}$, *and*

(vi) $P^{\rho}X^{\nabla}(X^{\rho})^{-1}$.

Proof. Suppose $X \in \mathbb{D}$ satisfies $W(X,X) \equiv 0$. Then by the same proof given in Theorem 104, $X^*(t)P(t)X^{\Delta}(t)$ is Hermitian for all $t \in \mathbb{T}^{\kappa}$. Evaluating this expression at $\rho(t)$, we see that

$$X^*(\rho(t))P(\rho(t))X^{\Delta}(\rho(t))$$

is Hermitian. But $X \in \mathbb{D}$, and therefore, $X^{\Delta\rho} = X^{\nabla}$ which gives the second result.

To get the third result, note that

$$(X^{\sigma})^*PX = \left(X + \mu X^{\Delta}\right)^* PX = X^*PX + \mu(X^{\Delta})^*PX,$$

which is Hermitian by earlier parts of this proof.

Similarly, to get the fourth result, note that

$$X^*P^{\rho}X^{\rho} = \left(X^{\rho} + \nu X^{\nabla}\right)^* P^{\rho}X^{\rho} = (X^{\rho})^*P^{\rho}X^{\rho} + \nu(X^{\nabla})^*P^{\rho}X^{\rho},$$

which is Hermitian by earlier parts of this proof.

Now, assume that, in addition, $X(t)$ is invertible for all $t \in \mathbb{T}^{\kappa}$. Then by earlier parts of this proof, we have that

$$(X^{\sigma})^*PX = X^*PX^{\sigma}. \tag{4.2}$$

Multiplication by $(X^{-1})^*$ on the left, and X^{-1} on the right gives

$$(X^{-1})^*(X^{\sigma})^*P = PX^{\sigma}X^{-1},$$

which shows that $PX^{\sigma}X^{-1}$ is Hermitian. Similarly, multiplying equation (4.2) on the left by $((X^{\sigma})^{-1})^*$, and on the right by $(X^{\sigma})^{-1}$, we get

$$PX(X^{\sigma})^{-1} = ((X^{\sigma})^{-1})^*X^*P,$$

which shows that $PX(X^{\sigma})^{-1}$ is Hermitian.

Furthermore, by Theorem 104, we have

$$X^*PX^{\Delta} = (X^{\Delta})^*PX.$$

Multiplying on the left by $(X^{-1})^*$, and on the right by X^{-1}, we get

$$PX^{\Delta}X^{-1} = (X^{-1})^*(X^{\Delta})^*P,$$

which shows that $PX^{\Delta}X^{-1}$ is Hermitian.

The remaining three results are derived similarly from the expressions

$$(X^\rho)^* P^\rho X^\nabla = (X^\nabla)^* P^\rho X^\rho \quad \text{and} \quad X^* P^\rho X^\rho = (X^\rho)^* P^\rho X.$$

□

Lemma 107. *The principal solution, X, of* (4.1) *at t_0 and the associated solution Y, of* (4.1) *at t_0 are normalized conjoined bases of* (4.1).

Proof. Note that

$$X^*(t_0)P(t_0)X^\Delta(t_0) = Y^*(t_0)P(t_0)Y^\Delta(t_0) = 0,$$

which is Hermitian. Thus by Theorem 104, X and Y are both prepared solutions of (4.1). It remains to show that $W(X,Y)(t) \equiv I$. We get

$$\begin{aligned} W(X,Y)(t_0) &= X^*(t_0)P(t_0)Y^\Delta(t_0) - [P(t_0)X^\Delta(t_0)]^*Y(t_0) \\ &= 0 - [P(t_0)P^{-1}(t_0)]^*(-I) \\ &= I. \end{aligned}$$

Then by Abel's formula we have $W(X,Y)(t) \equiv I$, as desired. □

Theorem 108 (Reduction of Order). *Assume that U is a prepared solution of* (4.1) *such that $U(t)$ is invertible for all $t \in \mathbb{T}$, and let*

$$V(t) := U(t) \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\, \Delta s.$$

Then V is a second solution of (4.1). *Moreover, U and V are normalized conjoined bases of* (4.1).

Proof. Assume U is a prepared solution of (4.1) such that $U(t)$ is invertible for all $t \in \mathbb{T}$. We will first show that V as defined above is a solution of (4.1). By Theorem 102, we need only show that $V \in \mathbb{D}$, and $W(U,V) \equiv C$ on $\mathbb{T}^\kappa$, where C is a constant matrix.

In the following calculations, we will suppress the argument t to make things easier to read, except in the integral where we include the arguments for the sake of clarity. Consider

$$\begin{aligned} W(U,V) &= U^*PV^\Delta - [PU^\Delta]^*V \\ &= U^*P\left[U^\sigma(U^*PU^\sigma)^{-1} + U^\Delta \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\, \Delta s\right] \\ &\quad -(U^\Delta)^*PU \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\, \Delta s \end{aligned}$$

$$\begin{aligned} &= I + U^*PU^\Delta \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s \\ &\quad -(U^\Delta)^*PU \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s \end{aligned}$$

Now U satisfies the conditions of Corollary 106, so $(U^\Delta)^*PU$ is Hermitian, and the last two terms therefore cancel. This gives

$$W(U,V) = I,$$

and we have that the Wronskian of U and V is a constant matrix, as desired.

To show that $V \in \mathbb{D}$, consider

$$\begin{aligned} V^\Delta &= U^\sigma(U^*PU^\sigma)^{-1} + U^\Delta \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s \\ &= U^\sigma(U^\sigma)^{-1}P^{-1}(U^*)^{-1} + U^\Delta \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s \\ &= P^{-1}(U^*)^{-1} + U^\Delta \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s, \end{aligned}$$

which is continuous. Then

$$\begin{aligned} [PV^\Delta]^\nabla &= \left[(U^*)^{-1} + PU^\Delta \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^\nabla \\ &= -(U^*)^{-1}(U^*)^\nabla((U^*)^\rho)^{-1} + [PU^\Delta]^\nabla \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s \\ &\quad +[PU^\Delta]^\rho \left[\int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^\nabla . \end{aligned}$$

The first two terms are clearly ld-continuous, but we need to simplify the third term further. By Corollary 27, we have that

$$\left[\int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^\nabla = \begin{cases} (U^*PU^\sigma)^{-1}(\rho(t)) & t \in \mathbb{T}_A \\ \lim_{s\to t^-} (U^*PU^\sigma)^{-1}(s) & t \in A. \end{cases}$$

Recall that A is the set of points which are left-dense and right-scattered. Thus if $t \in \mathbb{T}_A$, $\sigma(\rho(t)) = t$. This gives

$$\begin{aligned} \left[\int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^\nabla &= U^{-1}(\sigma(\rho(t)))P^{-1}(\rho(t))(U^*)^{-1}(\rho(t)) \\ &= U^{-1}(t)(P^\rho)^{-1}(t)((U^*)^\rho)^{-1}(t) \end{aligned}$$

for $t \in \mathbb{T}_A$. On the other hand, if $t \in A$, then t is left-dense and right-scattered, and we get

$$\begin{aligned}\left[\int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^\nabla &= \lim_{s\to t^-} (U^*PU^\sigma)^{-1}(s)\\ &= U^{-1}(t)P^{-1}(t)(U^*)^{-1}(t)\\ &= U^{-1}(t)(P^\rho)^{-1}(t)((U^*)^\rho)^{-1}(t),\end{aligned}$$

for $t \in A$. Looking at this again, we see that in either case, we get

$$\left[\int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^\nabla = U^{-1}(t)(P^\rho)^{-1}(t)((U^*)^\rho)^{-1}(t),$$

and thus,

$$[PU^\Delta]^\rho \left[\int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^\nabla = P^\rho U^{\Delta\rho} U^{-1}(P^\rho)^{-1}((U^*)^\rho)^{-1},$$

which is ld-continuous. So, $V \in \mathbb{D}$, and therefore by Theorem 102, V is a solution of (4.1).

To show that U and V are normalized conjoined bases, we must show that V is a prepared solution. Earlier calculations have already established that $W(U,V) = I$. Recall that by Theorem 104, V is a prepared solution if and only if V^*PV^Δ is Hermitian. Again suppressing the arguments, (except in the integral) we get

$$\begin{aligned}V^*PV^\Delta &= \left[\int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^* U^*P\Big[U^\sigma(U^\sigma)^{-1}P^{-1}(U^*)^{-1}\\ &\quad +U^\Delta \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\Big]\\ &= \left[\int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^*\\ &\quad + \left[\int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s\right]^* U^*PU^\Delta \int_{t_0}^{t} (U^*PU^\sigma)^{-1}(s)\,\Delta s.\end{aligned}$$

Now, since U is a prepared solution of (4.1), we have that U^*PU^Δ and U^*PU^σ are Hermitian. Therefore V^*PV^Δ is Hermitian, and thus V is a prepared solution of (4.1), which completes the proof. $\square$

4.3 Riccati Techniques

Next, we consider the matrix Riccati equation

$$RZ = 0, \quad \text{where} \quad RZ = Z^{\nabla} + Q(t) + (Z^{\rho})^{*}\left[P^{\rho}(t) + \nu(t)Z^{\rho}\right]^{-1}(Z^{\rho}). \tag{4.3}$$

Here, as with the self-adjoint matrix equation, we assume P and Q are Hermitian $n \times n$ matrices, $P(t)$ is invertible for all $t \in \mathbb{T}$, P is continuous, and Q is ld-continuous. We will take the domain of R, denoted $\mathbb{D}_R$, to be the set of all functions $Z : \mathbb{T}^{\kappa} \to \mathbb{R}$ such that Z is nabla-differentiable, $Z^{\nabla} : \mathbb{T}^{\kappa}_{\kappa} \to \mathbb{R}$ is ld-continuous and such that $P^{\rho}(t) + \nu(t)Z^{\rho}(t)$ is invertible for any $t \in \mathbb{T}^{\kappa}_{\kappa}$. A function $Z \in \mathbb{D}_R$ is said to be a solution of (4.3) on $\mathbb{T}^{\kappa}$ provided $RZ(t) = 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

The following theorem shows us how the self-adjoint matrix equation, (4.1) and the Riccati matrix equation are related.

Theorem 109. *Assume $X \in \mathbb{D}$ satisfies $W(X, X) \equiv 0$, and that $X(t)$ is invertible for all $t \in \mathbb{T}$. Let Z be defined by the Riccati substitution*

$$Z(t) = P(t)X^{\Delta}(t)X^{-1}(t), \tag{4.4}$$

for $t \in \mathbb{T}^{\kappa}$. Then $Z \in \mathbb{D}_R$ is Hermitian, and

$$LX(t) = [RZ(t)]\, X(t)$$

for $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Proof. Suppose X is as described in the theorem, and define Z by the Riccati substitution, (4.4). Then, by Corollary 106, Z is Hermitian. To see that $Z \in \mathbb{D}_R$, note that $X \in \mathbb{D}$, which implies PX^{Δ} and X^{-1} are nabla-differentiable, so Z is nabla-differentiable. Furthermore,

$$\begin{aligned}
Z^{\nabla}(t) &= [PX^{\Delta}X^{-1}]^{\nabla}(t) \\
&= [PX^{\Delta}]^{\nabla}(t)(X^{\rho})^{-1}(t) + [PX^{\Delta}](t)[X^{-1}]^{\nabla}(t) \\
&= [PX^{\Delta}]^{\nabla}(t)(X^{\rho})^{-1}(t) - P(t)X^{\Delta}(t)X^{-1}(t)X^{\nabla}(t)(X^{\rho})^{-1}(t) \\
&= [PX^{\Delta}]^{\nabla}(t)(X^{\rho})^{-1}(t) - P(t)X^{\Delta}(t)X^{-1}(t)X^{\Delta\rho}(t)(X^{\rho})^{-1}(t),
\end{aligned}$$

which is ld-continuous. Next, note that

$$\begin{aligned}
P^{\rho}(t) + \nu(t)Z^{\rho}(t) &= P^{\rho}(t) + \nu(t)[PX^{\Delta}X^{-1}]^{\rho}(t) \\
&= P^{\rho}(t)[I + \nu(t)X^{\Delta\rho}(t)(X^{\rho})^{-1}(t)] \\
&= P^{\rho}(t)[X^{\rho}(t) + \nu(t)X^{\nabla}(t)](X^{\rho})^{-1}(t) \\
&= P^{\rho}(t)X(t)(X^{\rho})^{-1}(t),
\end{aligned}$$

which is invertible, and hence $Z \in \mathbb{D}_R$. Finally, to complete the proof, we need to show

$$LX(t) = [RZ(t)]\, X(t)$$

for $t \in \mathbb{T}^\kappa_\kappa$. Suppressing the arguments, we get

$$\begin{aligned}
RZ &= Z^\nabla + Q + (Z^\rho)^*[P^\rho + \nu Z^\rho]^{-1}(Z^\rho) \\
&= Z^\nabla + Q + Z^\rho[P^\rho + P^\rho(\nu X^{\Delta\rho})(X^\rho)^{-1}]^{-1}(Z^\rho) \\
&= Z^\nabla + Q + Z^\rho[P^\rho[I + (X - X^\rho)(X^\rho)^{-1}]]^{-1}(Z^\rho) \\
&= Z^\nabla + Q + Z^\rho[I + X(X^\rho)^{-1} - I]^{-1}(P^\rho)^{-1}(Z^\rho) \\
&= Z^\nabla + Q + Z^\rho[X(X^\rho)^{-1}]^{-1}(P^\rho)^{-1}Z^\rho \\
&= Z^\nabla + Q + Z^\rho X^\rho X^{-1}(P^\rho)^{-1}Z^\rho \\
&= [PX^\Delta X^{-1}]^\nabla + Q + (PX^\Delta X^{-1})^\rho X^\rho X^{-1}(P^\rho)^{-1}(PX^\Delta X^{-1})^\rho \\
&= [PX^\Delta]^\nabla X^{-1} + [PX^\Delta]^\rho (X^{-1})^\nabla + Q \\
&\quad + P^\rho X^{\Delta\rho}(X^\rho)^{-1}X^\rho X^{-1}(P^\rho)^{-1}P^\rho X^{\Delta\rho}(X^\rho)^{-1} \\
&= [PX^\Delta]^\nabla X^{-1} - [PX^\Delta]^\rho[X^{-1}X^\nabla(X^\rho)^{-1}] + Q + P^\rho X^{\Delta\rho}X^{-1}X^{\Delta\rho}(X^\rho)^{-1} \\
&= [PX^\Delta]^\nabla X^{-1} + QXX^{-1} \\
&= [LX]X^{-1}.
\end{aligned}$$

Multiplying on the right by X then gives the desired result. ☐

Theorem 110. *The self-adjoint matrix equation,* (4.1)*, has a prepared solution, X, such that $X(t)$ is invertible for all $t \in \mathbb{T}$ if and only if the Riccati matrix equation,* (4.3)*, has a Hermitian solution Z. In this case, X and Z are related by the Riccati substitution* (4.4)*.*

Proof. First, suppose that X is a prepared solution of (4.1) such that $X(t)$ is invertible for all $t \in \mathbb{T}$, and define $Z(t)$ by the Riccati substitution, (4.4). Then by Theorem 109, $Z \in \mathbb{D}_R$ is Hermitian, and

$$RZ(t) = [LX](t)X^{-1}(t) = 0.$$

Conversely, suppose the Riccati matrix equation, (4.3) has a Hermitian solution Z. Then $-(P^\rho)^{-1}Z^\rho$ is ld-continous. Furthermore, $Z \in \mathbb{D}_R$, so

$$I + \nu(P^\rho)^{-1}Z^\rho = (P^\rho)^{-1}[P^\rho + \nu Z^\rho]$$

is invertible, and hence $-(P^\rho)^{-1}Z^\rho \in \mathcal{R}_\nu$. Then, by Lemma 95, $P^{-1}Z \in \mathcal{R}$. Now, let X be the solution of the initial value problem

$$X^\Delta = P^{-1}(t)Z(t)X, \quad X(t_0) = I,$$

or, in other words, let $X = e_{P^{-1}Z}(t, t_0)$. Note that although $P^{-1}Z$ is only defined on T^κ, X is defined on all of $\mathbb{T}$. We immediately see that $X(t)$ is invertible for all $t \in \mathbb{T}$,

that X is delta-differentiable, and that X^Δ is continuous. Next, consider

$$\begin{aligned}
[P(t)X^\Delta(t)]^\nabla &= [P(t)P^{-1}(t)Z(t)X(t)]^\nabla \\
&= [Z(t)X(t)]^\nabla \\
&= Z^\nabla(t)X^\rho(t) + Z(t)X^\nabla(t) \\
&= Z^\nabla(t)X^\rho(t) + Z(t)X^{\Delta\rho}(t),
\end{aligned}$$

which is ld-continuous on $\mathbb{T}^\kappa_\kappa$. Hence $X \in \mathbb{D}$. Suppressing the arguments,

$$\begin{aligned}
W(X,X) &= X^*PX^\Delta - [PX^\Delta]^*X \\
&= X^*P[P^{-1}ZX] - [PP^{-1}ZX]^*X \\
&= X^*ZX - X^*ZX \\
&= 0.
\end{aligned}$$

Furthermore, $Z(t) = P(t)X^\Delta(t)X^{-1}(t)$. Hence the conditions of Theorem 109 are met, and we have

$$LX(t) = [RZ](t)X^{-1}(t) = 0.$$

Therefore X is a prepared solution of (4.1). $\square$

Next, we wish to develop Jacobi's condition for the self-adjoint matrix equation (4.1). Toward that end, we want to establish Picone's Identity, which will be crucial. The calculations required to establish Picone's Identity are quite lengthy, however, so we break things up by first establishing a computational lemma.

Lemma 111 (Completing the Square). *Let X be a prepared solution of* (4.1) *such that $X(t)$ is invertible for all $t \in \mathbb{T}$, and define Z via the Riccati substitution* (4.4). *Fix $t \in \mathbb{T}^\kappa$, let u be an n-dimensional vector function which is nabla-differentiable at t, and let $D = X^\rho X^{-1}(P^\rho)^{-1}$. Then at t, we have*

$$(u^*Zu)^\nabla = (u^\nabla)^*P^\rho u^\nabla - u^*Qu - [P^\rho u^\nabla - Z^\rho u^\rho]^*D[P^\rho u^\nabla - Z^\rho u^\rho].$$

Proof. Let X, Z, and u be as described in the lemma. Note that Theorem 110 applies and thus Z is Hermitian, and is a solution of (4.3). Furthermore, on $\mathbb{T}^\kappa$, we have

$$\begin{aligned}
P^\rho + \nu Z^\rho &= P^\rho + \nu\left[PX^\Delta X^{-1}\right]^\rho \\
&= P^\rho + \nu P^\rho X^{\Delta\rho}(X^\rho)^{-1} \\
&= P^\rho\left[I + \nu X^\nabla (X^\rho)^{-1}\right] \\
&= P^\rho\left[I + (X - X^\rho)(X^\rho)^{-1}\right] \\
&= P^\rho\left[I + X(X^\rho)^{-1} - I\right] \\
&= P^\rho X(X^\rho)^{-1}.
\end{aligned}$$

Additionally, note that the Riccati substitution tells us that

$$Z(t) = P(t)X^{\Delta}(t)X^{-1}(t),$$

and therefore,

$$\begin{aligned} Z^{\rho}(t)X^{\rho}(t) &= P^{\rho}(t)X^{\Delta\rho}(t)(X^{\rho})^{-1}(t)X^{\rho}(t) \\ &= P^{\rho}(t)X^{\nabla}(t). \end{aligned}$$

Now fix $t \in \mathbb{T}$ such that u is nabla-differentiable at t. Then at t, we have

$$\begin{aligned}
&(u^*Zu)^{\nabla} + [P^{\rho}u^{\nabla} - Z^{\rho}u^{\rho}]^* X^{\rho}X^{-1}(P^{\rho})^{-1}[P^{\rho}u^{\nabla} - Z^{\rho}u^{\rho}] \\
&\quad = (u^*)(Zu)^{\nabla} + (u^{\nabla})^*(Zu)^{\rho} + (P^{\rho}u^{\nabla})^* X^{\rho}X^{-1}(P^{\rho})^{-1}(P^{\rho}u^{\nabla}) \\
&\qquad - (P^{\rho}u^{\nabla})^* X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} - (Z^{\rho}u^{\rho})^* X^{\rho}X^{-1}(P^{\rho})^{-1}(P^{\rho}u^{\nabla}) \\
&\qquad + (Z^{\rho}u^{\rho})^* X^{\rho}X^{-1}(P^{\rho})^{-1}(Z^{\rho}u^{\rho}) \\
&\quad = u^*[Z^{\nabla}u + Z^{\rho}u^{\nabla}] + (u^{\nabla})^* Z^{\rho}u^{\rho} + (u^{\nabla})^* P^{\rho}X^{\rho}X^{-1}u^{\nabla} \\
&\qquad - (u^{\nabla})^* P^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} - (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}u^{\nabla} \\
&\qquad + (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} \\
&\quad = u^*\left[-Q - Z^{\rho}[P^{\rho} + \nu Z^{\rho}]^{-1}Z^{\rho}\right]u + u^*Z^{\rho}u^{\nabla} + (u^{\nabla})^* Z^{\rho}u^{\rho} \\
&\qquad + (u^{\nabla})^* P^{\rho}X^{\rho}X^{-1}u^{\nabla} - (u^{\nabla})^* P^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} \\
&\qquad - (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}u^{\nabla} + (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} \\
&\quad = -u^*Qu - u^*Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u + u^*Z^{\rho}u^{\nabla} \\
&\qquad + (u^{\nabla})^* Z^{\rho}u^{\rho} + (u^{\nabla})^* P^{\rho}[X - \nu X^{\nabla}]X^{-1}u^{\nabla} \\
&\qquad - (u^{\nabla})^* P^{\rho}[X - \nu X^{\nabla}]X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} - (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}u^{\nabla} \\
&\qquad + (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} \\
&\quad = -u^*Qu - u^*Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u + u^*Z^{\rho}u^{\nabla} + (u^{\nabla})^* Z^{\rho}u^{\rho} \\
&\qquad + (u^{\nabla})^* P^{\rho}u^{\nabla} - \nu(u^{\nabla})^* P^{\rho}X^{\nabla}X^{-1}u^{\nabla} - (u^{\nabla})^* Z^{\rho}u^{\rho} \\
&\qquad + \nu(u^{\nabla})^* P^{\rho}X^{\nabla}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} - (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}u^{\nabla} \\
&\qquad + (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} \\
&\quad = (u^{\nabla})^* P^{\rho}u^{\nabla} - u^*Qu - u^*Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u + u^*Z^{\rho}u^{\nabla} \\
&\qquad - \nu(u^{\nabla})^* P^{\rho}X^{\nabla}X^{-1}u^{\nabla} + \nu(u^{\nabla})^* P^{\rho}X^{\nabla}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} \\
&\qquad - (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}u^{\nabla} + (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} \\
&\quad = (u^{\nabla})^* P^{\rho}u^{\nabla} - u^*Qu - u^*Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u + u^*Z^{\rho}u^{\nabla} \\
&\qquad - \nu(u^{\nabla})^* Z^{\rho}X^{\rho}X^{-1}u^{\nabla} + \nu(u^{\nabla})^* Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} \\
&\qquad - (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}u^{\nabla} + (u^{\rho})^* Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u^{\rho} \\
&\quad = (u^{\nabla})^* P^{\rho}u^{\nabla} - u^*Qu - u^*Z^{\rho}X^{\rho}X^{-1}(P^{\rho})^{-1}Z^{\rho}u + u^*Z^{\rho}u^{\nabla}
\end{aligned}$$

$$
\begin{aligned}
&-[u^\rho+\nu u^\nabla]^* Z^\rho X^\rho X^{-1}u^\nabla + [u^\rho+\nu u^\nabla]^* Z^\rho X^\rho X^{-1}(P^\rho)^{-1}Z^\rho u^\rho \\
=\;& (u^\nabla)^* P^\rho u^\nabla - u^*Qu - u^*Z^\rho X^\rho X^{-1}(P^\rho)^{-1}Z^\rho u + u^*Z^\rho u^\nabla \\
&- u^*Z^\rho X^\rho X^{-1}u^\nabla + u^*Z^\rho X^\rho X^{-1}(P^\rho)^{-1}Z^\rho[u-\nu u^\nabla] \\
=\;& (u^\nabla)^* P^\rho u^\nabla - u^*Qu - u^*Z^\rho X^\rho X^{-1}(P^\rho)^{-1}Z^\rho u + u^*Z^\rho u^\nabla \\
&- u^*Z^\rho X^\rho X^{-1}u^\nabla + u^*Z^\rho X^\rho X^{-1}(P^\rho)^{-1}Z^\rho u - \nu u^*Z^\rho X^\rho X^{-1}(P^\rho)^{-1}Z^\rho u^\nabla \\
=\;& (u^\nabla)^* P^\rho u^\nabla - u^*Qu \\
&+ u^*\left[Z^\rho - Z^\rho X^\rho X^{-1} - \nu Z^\rho X^\rho X^{-1}(P^\rho)^{-1}Z^\rho\right]u^\nabla.
\end{aligned}
$$

We now claim that the last term cancels, which will complete the proof. Looking only at the center expression of the last term, we have

$$
\begin{aligned}
Z^\rho - & Z^\rho X^\rho X^{-1} - \nu Z^\rho X^\rho X^{-1}(P^\rho)^{-1}Z^\rho \\
&= P^\rho X^{\Delta\rho}(X^\rho)^{-1} - P^\rho X^{\Delta\rho}(X^\rho)^{-1}X^\rho X^{-1} \\
&\quad - \nu(P^\rho X^{\Delta\rho}(X^\rho)^{-1})X^\rho X^{-1}(P^\rho)^{-1}P^\rho X^{\Delta\rho}(X^\rho)^{-1} \\
&= P^\rho X^\nabla (X^\rho)^{-1} - P^\rho X^\nabla X^{-1} - \nu P^\rho X^\nabla X^{-1}X^\nabla (X^\rho)^{-1} \\
&= P^\rho X^\nabla\left[(X^\rho)^{-1} - X^{-1} - \nu(X^{-1}X^\nabla(X^\rho)^{-1}\right] \\
&= P^\rho X^\nabla\left[(X^\rho)^{-1} - X^{-1} - X^{-1}[X - X^\rho](X^\rho)^{-1}\right] \\
&= P^\rho X^\nabla\left[(X^\rho)^{-1} - X^{-1} - [I - X^{-1}X^\rho](X^\rho)^{-1}\right] \\
&= P^\rho X^\nabla\left[(X^\rho)^{-1} - X^{-1} - [(X^\rho)^{-1} - X^{-1}]\right] \\
&= P^\rho X^\nabla(0) \\
&= 0.
\end{aligned}
$$

So, the last term does, in fact cancel, which yields the result. $\square$

The above lemma will be extremely useful in establishing the following result.

Theorem 112 (Picone's Identity). *Let $\alpha \in \mathbb{C}^n$ and assume X and Y are normalized conjoined bases of* (4.1) *such that $X(t)$ is invertible for all $t \in \mathbb{T}$. Put*

$$
Z = PX^\Delta X^{-1} \quad \text{and} \quad D = X^\rho X^{-1}(P^\rho)^{-1} \quad \text{on } \mathbb{T}^\kappa.
$$

Furthermore, fix $t \in \mathbb{T}^\kappa$, and assume u is an n-dimensional vector-valued function such that u is nabla-differentiable at t. Then at t, we have

$$
\begin{aligned}
&(u^*Zu + \alpha^*X^{-1}u + u^*(X^{-1})^*\alpha - \alpha^*X^{-1}Y\alpha)^\nabla - (u^\nabla)^*P^\rho u^\nabla + u^*Qu \\
&\quad = -[P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^*\alpha]^* D[P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^*\alpha].
\end{aligned}
$$

Proof. We will look at each of the terms individually, then put it all together to get the desired result.

First, consider

$$\begin{aligned}(X^{-1}Y)^{\nabla} &= (X^{-1})^{\nabla}Y^{\rho} + X^{-1}Y^{\nabla}\\ &= -X^{-1}X^{\nabla}(X^{\rho})^{-1}Y^{\rho} + X^{-1}Y^{\nabla}\\ &= -X^{-1}(P^{\rho})^{-1}P^{\rho}X^{\nabla}(X^{\rho})^{-1}Y^{\rho} + X^{-1}Y^{\nabla}.\end{aligned}$$

Now, by Corollary 106, $P^{\rho}X^{\nabla}(X^{\rho})^{-1}$ is Hermitian, so we see that

$$P^{\rho}X^{\nabla}(X^{\rho})^{-1} = ((X^{\rho})^{-1})^{*}(X^{\nabla})^{*}P^{\rho}.$$

Substitution then gives

$$\begin{aligned}(X^{-1}Y)^{\nabla} &= -X^{-1}(P^{\rho})^{-1}((X^{\rho})^{-1})^{*}(X^{\nabla})^{*}P^{\rho}Y^{\rho} + X^{-1}(P^{\rho})^{-1}((X^{\rho})^{-1})^{*}(X^{\rho})^{*}P^{\rho}Y^{\nabla}\\ &= X^{-1}(P^{\rho})^{-1}((X^{\rho})^{-1})^{*}\left[-(X^{\nabla})^{*}P^{\rho}Y^{\rho} + (X^{\rho})^{*}P^{\rho}Y^{\nabla}\right]\\ &= X^{-1}(P^{\rho})^{-1}((X^{\rho})^{-1})^{*}\left[X^{*}[PY^{\Delta}] - [PX^{\Delta}]^{*}Y\right]^{\rho}\\ &= X^{-1}(P^{\rho})^{-1}((X^{\rho})^{-1})^{*}W^{\rho}(X,Y)\\ &= X^{-1}(P^{\rho})^{-1}((X^{\rho})^{-1})^{*}.\end{aligned}$$

Next, look at

$$\begin{aligned}(\alpha^{*}X^{-1}u)^{\nabla} &= \alpha^{*}(X^{-1})^{\nabla}u^{\rho} + \alpha^{*}X^{-1}u^{\nabla}\\ &= \alpha^{*}X^{-1}X^{\nabla}(X^{\rho})^{-1}u^{\rho} + \alpha^{*}X^{-1}u^{\nabla}\\ &= -\alpha^{*}(X^{\rho})^{-1}X^{\rho}X^{-1}(P^{\rho})^{-1}P^{\rho}X^{\nabla}(X^{\rho})^{-1}u^{\rho}\\ &\quad + \alpha^{*}(X^{\rho})^{-1}X^{\rho}X^{-1}(P^{\rho})^{-1}P^{\rho}u^{\nabla}\\ &= \alpha^{*}(X^{\rho})^{-1}\left[DP^{\rho}u^{\nabla} - DP^{\rho}X^{\nabla}(X^{\rho})^{-1}u^{\rho}\right]\\ &= \alpha^{*}(X^{\rho})^{-1}\left[DP^{\rho}u^{\nabla} - DZ^{\rho}X^{\rho}(X^{\rho})^{-1}u^{\rho}\right]\\ &= \alpha^{*}(X^{\rho})^{-1}D\left[P^{\rho}u^{\nabla} - Z^{\rho}u^{\rho}\right].\end{aligned}$$

Now, note that since $P^{\rho}X(X^{\rho})^{-1}$ is Hermitian, $D = [P^{\rho}X(X^{\rho})^{-1}]^{-1}$ is Hermitian as well. This gives

$$\begin{aligned}(u^{*}(X^{-1})^{*}\alpha)^{\nabla} &= ((\alpha^{*}X^{-1}u)^{\nabla})^{*}\\ &= \left[\alpha^{*}(X^{\rho})^{-1}D\left[P^{\rho}u^{\nabla} - Z^{\rho}u^{\rho}\right]\right]^{*}\\ &= \left[P^{\rho}u^{\nabla} - Z^{\rho}u^{\rho}\right]^{*}D((X^{\rho})^{-1})^{*}\alpha\end{aligned}$$

Now, putting this all together, and applying Lemma 111, we see that

$$\begin{aligned}&(u^{*}Zu + \alpha^{*}X^{-1}u + u^{*}(X^{-1})^{*}\alpha - \alpha^{*}X^{-1}Y\alpha)^{\nabla} - (u^{\nabla})^{*}P^{\rho}u^{\nabla} + u^{*}Qu\\ &\quad = -[P^{\rho}u^{\nabla} - Z^{\rho}u^{\rho}]^{*}D[P^{\rho}u^{\nabla} - Z^{\rho}u^{\rho}]\\ &\qquad + (\alpha^{*}X^{-1}u)^{\nabla} + (u^{*}(X^{-1})^{*}\alpha)^{\nabla} - \alpha^{*}(X^{-1}Y)^{\nabla}\alpha\end{aligned}$$

$$
\begin{aligned}
&= -[P^\rho u^\nabla - Z^\rho u^\rho]^* D [P^\rho u^\nabla - Z^\rho u^\rho] \\
&\quad + \alpha^* (X^\rho)^{-1} D \left[P^\rho u^\nabla - Z^\rho u^\rho\right] + \left[P^\rho u^\nabla - Z^\rho u^\rho\right]^* D((X^\rho)^{-1})^* \alpha \\
&\quad\quad - \alpha^* X^{-1} (P^\rho)^{-1} ((X^\rho)^{-1})^* \alpha \\
&= -[P^\rho u^\nabla - Z^\rho u^\rho]^* D [P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^* \alpha] \\
&\quad + \alpha^* (X^\rho)^{-1} D \left[P^\rho u^\nabla - Z^\rho u^\rho\right] - \alpha^* (X^\rho)^{-1} X^\rho X^{-1} (P^\rho)^{-1} ((X^\rho)^{-1})^* \alpha \\
&= -[P^\rho u^\nabla - Z^\rho u^\rho]^* D [P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^* \alpha] \\
&\quad + \alpha^* (X^\rho)^{-1} D \left[P^\rho u^\nabla - Z^\rho u^\rho\right] - \alpha^* (X^\rho)^{-1} D ((X^\rho)^{-1})^* \alpha \\
&= -[P^\rho u^\nabla - Z^\rho u^\rho]^* D [P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^* \alpha] \\
&\quad + \alpha^* (X^\rho)^{-1} D [P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^* \alpha] \\
&= -[P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^* \alpha]^* D [P^\rho u^\nabla - Z^\rho u^\rho ((X^\rho)^{-1})^* \alpha],
\end{aligned}
$$

which completes the proof. □

Remark 113. In Lemma 111 and Theorem 112, we assumed that X was a prepared solution of (4.1) and that $X(t)$ was invertible for all $t \in \mathbb{T}$. In the work that follows, we are going to consider a prepared solution, X, of (4.1) on an interval $[a, b]$, but we are going to allow the possibility that $X(a)$ is singular. Of course, if $X(a)$ is singular, then the identities are no longer valid at $t = a$. A careful review of the proofs shows that there may also be problems at $t = \sigma(a)$. For $t \in (\sigma(a), b]$, however, the identities hold. Therefore, in the proofs of the next results, we will take care to apply Lemma 111 and Theorem 112 only on $(\sigma(a), b]$.

Definition 114. Let $\mathcal{F}$ be the quadratic functional defined by

$$\mathcal{F}[u] = \int_a^b \left[(u^\nabla)^* P^\rho u^\nabla - u^* Q u\right](t)\, \nabla t.$$

We say $\mathcal{F}$ is *positive definite* provided $\mathcal{F}[u] \geq 0$ for all $u \in \mathrm{C}^1_{\mathrm{pld}}([a, b], \mathbb{R}^n)$ such that $u(a) = u(b) = 0$, and $\mathcal{F}[u] = 0$ if and only if $u \equiv 0$.

Here $\mathrm{C}^1_{\mathrm{pld}}([a, b])$ denotes the set of continuous n-dimensional vector-valued functions whose nabla-derivatives are piecewise ld-continuous.

Definition 115. A prepared solution, X, of (4.1) is said to have no focal points on $(a, b]$ provided

$$X \text{ is invertible on } (a, b], \quad \text{and} \quad X^\rho X^{-1} (P^\rho)^{-1} \geq 0 \text{ on } (a, b].$$

We continue to work towards Jacobi's condition. The next result establishes a sufficient condition for the quadratic functional $\mathcal{F}$ to be positive definite.

Theorem 116 (Sufficient Condition for Positive Definiteness of $\mathcal{F}$). *If there exist normalized conjoined bases X and Y of* (4.1) *such that X has no focal points in $(a, b]$, then $\mathcal{F}$ is positive definite.*

Proof. Let X and Y be as described in the theorem, and let $D = X^\rho X^{-1}(P^\rho)^{-1}$ on $(a,b]$. Fix $u \in \mathrm{C}^1_{\mathrm{pld}}([a,b])$ with $u(a) = u(b) = 0$.

We will consider two cases, a right-scattered and a right-dense. So, first, assume a is right-scattered. Then, applying Lemma 111, we get

$$\begin{aligned}
\mathcal{F}[u] &= \int_a^b \left[(u^\nabla)^* P^\rho u^\nabla - u^* Q u\right](t)\ \nabla t \\
&= \int_a^{\sigma(a)} \left[(u^\nabla)^* P^\rho u^\nabla - u^* Q u\right](t)\ \nabla t + \int_{\sigma(a)}^b \left[(u^\nabla)^* P^\rho u^\nabla - u^* Q u\right](t)\ \nabla t \\
&= [(u^\nabla)^* P^\rho u^\nabla - u^* Q u](\sigma(a))\nu(\sigma(a)) \\
&\quad + \int_{\sigma(a)}^b \left[(u^* Z u)^\nabla + [P^\rho u^\nabla - Z^\rho u^\rho]^* D [P^\rho u^\nabla - Z^\rho u^\rho]\right](t)\ \nabla t \\
&= [(\nu u^\nabla)^* P^\rho (\nu u^\nabla)\nu^{-1} - \nu u^* Q u](\sigma(a)) + (u^* Z u)(b) - (u^* Z u)(\sigma(a)) \\
&\quad + \int_{\sigma(a)}^b [P^\rho u^\nabla - Z^\rho u^\rho]^* D [P^\rho u^\nabla - Z^\rho u^\rho](t)\ \nabla t
\end{aligned}$$

Recall that X has no focal points in $(a,b]$, and therefore $D \geq 0$ on $(\sigma(a), b]$. Additionally, $u(b) = 0$, so one of our terms cancels, and we get

$$\begin{aligned}
\mathcal{F}[u] &\geq \left\{(\nu u^\nabla)^* P^\rho (\nu u^\nabla)\nu^{-1} - \nu u^* Q u - (u^* Z u)\right\}(\sigma(a)) \\
&= \left\{[u - u^\rho]^* P^\rho [u - u^\rho]\nu^{-1} - \nu u^* Q X X^{-1} u - u^* Z u\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho u \nu^{-1} + \nu u^* \left[P X^\Delta\right]^\nabla X^{-1} u - u^* P X^\Delta X^{-1} u\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho u \nu^{-1} + u^* [P X^\Delta - P^\rho X^{\Delta\rho}] X^{-1} u - u^* P X^\Delta X^{-1} u\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho u \nu^{-1} + u^* P X^\Delta X^{-1} u - u^* P^\rho X^\nabla X^{-1} u - u^* P X^\Delta X^{-1} u\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho u \nu^{-1} - u^* P^\rho X^\nabla X^{-1} u\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho [u \nu^{-1} - X^\nabla X^{-1} u]\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho [I - \nu X^\nabla X^{-1}] u \nu^{-1}\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho [I - [X - X^\rho] X^{-1}] u \nu^{-1}\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho [I - I + X^\rho X^{-1}] u \nu^{-1}\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho X^\rho X^{-1} u \nu^{-1}\right\}(\sigma(a)) \\
&= \left\{u^* P^\rho X^\rho X^{-1} (P^\rho)^{-1} P^\rho u \nu^{-1}\right\}(\sigma(a)) \\
&= \{(P^\rho u)^* D (P^\rho u)\}(\sigma(a))\nu^{-1}(\sigma(a)) \\
&\geq 0.
\end{aligned}$$

Clearly, $\mathcal{F}[0] = 0$, therefore to complete the first case, we must show that $\mathcal{F}[u] =$

$0 \implies u \equiv 0$. So, assume $\mathcal{F}[u] = 0$. By the work we did above, we have

$$\mathcal{F}[u] = \left\{(\nu u^{\nabla})^* P^{\rho}(\nu u^{\nabla})\nu^{-1} - \nu u^* Q u - (u^* Z u)\right\}(\sigma(a)) + \int_{\sigma(a)}^{b} [P^{\rho} u^{\nabla} - Z^{\rho} u^{\rho}]^* D [P^{\rho} u^{\nabla} - Z^{\rho} u^{\rho}](t)\ \nabla t,$$

and

$$\left\{(\nu u^{\nabla})^* P^{\rho}(\nu u^{\nabla})\nu^{-1} - \nu u^* Q u - (u^* Z u)\right\}(\sigma(a)) \geq 0.$$

Therefore, since $\mathcal{F}[u] = 0$, we must have

$$\left\{(\nu u^{\nabla})^* P^{\rho}(\nu u^{\nabla})\nu^{-1} - \nu u^* Q u - (u^* Z u)\right\}(\sigma(a)) = 0,$$

and

$$\int_{\sigma(a)}^{b} [P^{\rho} u^{\nabla} - Z^{\rho} u^{\rho}]^* D [P^{\rho} u^{\nabla} - Z^{\rho} u^{\rho}](t)\ \nabla t = 0.$$

Thus

$$P^{\rho} u^{\nabla} - Z^{\rho} u^{\rho} = 0$$

on $(\sigma(a), b]$. Hence on $(\sigma(a), b]$, we have

$$\begin{aligned} 0 &= P^{\rho} u^{\nabla} - Z^{\rho} u^{\rho} \\ &= P^{\rho} u^{\nabla} - P^{\rho} X^{\Delta\rho} (X^{\rho})^{-1} u^{\rho} \\ &= P^{\rho} [u^{\nabla} - X^{\nabla} (X^{\rho})^{-1} u^{\rho}]. \end{aligned}$$

Since P^{ρ} is invertible, we then get

$$\begin{aligned} 0 &= u^{\nabla} - X^{\nabla} (X^{\rho})^{-1} u^{\rho} \\ &= u^{\nabla} - X^{\nabla} (X^{\rho})^{-1} [u - \nu u^{\nabla}] \\ &= u^{\nabla} - X^{\nabla} (X^{\rho})^{-1} u + \nu X^{\nabla} (X^{\rho})^{-1} u^{\nabla}. \end{aligned}$$

Rearranging this, we see that

$$\begin{aligned} X^{\nabla} (X^{\rho})^{-1} u &= u^{\nabla} + \nu X^{\nabla} (X^{\rho})^{-1} u^{\nabla} \\ &= [I + \nu X^{\nabla} (X^{\rho})^{-1}] u^{\nabla} \\ &= [I + [X - X^{\rho}] (X^{\rho})^{-1}] u^{\nabla} \\ &= [I + X (X^{\rho})^{-1} - I] u^{\nabla} \\ &= X (X^{\rho})^{-1} u^{\nabla}. \end{aligned}$$

And finally, we have

$$u^{\nabla} = X^{\rho} X^{-1} X^{\nabla} (X^{\rho})^{-1} u$$

on $(\sigma(a), b]$. Now $X^\rho X^{-1} X^\nabla (X^\rho)^{-1}$ is ld-continuous, and

$$\begin{aligned} I - \nu[X^\rho X^{-1} X^\nabla (X^\rho)^{-1}] &= I - X^\rho X^{-1}[X - X^\rho](X^\rho)^{-1} \\ &= I - X^\rho X^{-1} X (X^\rho)^{-1} + X^\rho X^{-1} X^\rho (X^\rho)^{-1} \\ &= I - I + X^\rho X^{-1} \\ &= X^\rho X^{-1}, \end{aligned}$$

which is invertible for $t \in (\sigma(a), b]$. Thus $X^\rho X^{-1} X^\nabla (X^\rho)^{-1} \in \mathcal{R}_\nu$, and therefore the initial value problem

$$u^\nabla = X^\rho X^{-1} X^\nabla (X^\rho)^{-1} u, \quad u(b) = 0$$

has a unique solution (the trivial solution). Thus $u(t) = 0$ for $t \in (\sigma(a), b]$. Moreover, $u(a) = 0$. So, $u(t) = 0$ on $[a, b]$ except possibly at $\sigma(a)$. Now if $\sigma(a)$ is right-dense, then by continuity, $u(\sigma(a)) = 0$. The only remaining possibility is that $\sigma(a)$ is right-scattered, and

$$u(t) = \begin{cases} 0 & t \neq \sigma(a) \\ c & t = \sigma(a). \end{cases}$$

In this case,

$$u^\nabla(t) = \begin{cases} 0 & t \notin \{\sigma(a), \sigma^2(a)\} \\ \frac{c}{\nu(\sigma(a))} & t = \sigma(a) \\ \frac{-c}{\nu(\sigma^2(a))} & t = \sigma^2(a). \end{cases}$$

But $\sigma^2(a) \in (\sigma(a), b]$, so we have

$$\frac{-c}{\nu(\sigma^2(a))} = u^\nabla(\sigma^2(a)) = [X^\rho X^{-1} X^\nabla (X^\rho)^{-1}](\sigma^2(a)) u(\sigma^2(a)) = 0.$$

and thus $c = u(\sigma(a)) = 0$. This gives $u(t) \equiv 0$ on $[a, b]$, and therefore $\mathcal{F} > 0$, which completes the proof in the case where a is right-scattered.

We now turn our attention to the second case. Assume a is right-dense. Select a decreasing sequence, $\{a_m\}_{m=1}^\infty$, of points in $\mathbb{T}$ such that $\lim_{m\to\infty} a_m = a$, and for $m \in \mathbb{N}$, let $\alpha_m = -\left((X^\Delta)^* P u\right)(a_m)$. Note that as X^Δ, P, and u are continuous, we have

$$\lim_{m\to\infty} \alpha_m = -((X^\Delta)^* P u)(a) = 0.$$

Now applying Theorem 112 with $\alpha = \alpha_m$, we have

$$\begin{aligned} &\int_{a_m}^b \left[(u^\nabla)^* P^\rho u^\nabla - u^* Q u\right](t)\, \nabla t \\ &\quad = \int_{a_m}^b (u^* Z u + \alpha_m^* X^{-1} u + u^* (X^{-1})^* \alpha_m - \alpha_m^* X^{-1} Y \alpha_m)^\nabla\, \nabla t \end{aligned}$$

$$
\begin{aligned}
&+\int_{a_m}^{b}[P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^*\alpha_m]^* D[P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^*\alpha_m]\ \nabla t\\
&\geq \int_{a_m}^{b}(u^*Zu + \alpha_m^* X^{-1}u + u^*(X^{-1})^*\alpha_m - \alpha_m^* X^{-1}Y\alpha_m)^\nabla\ \nabla t\\
&= (\alpha_m^* X^{-1}Y\alpha_m)(b) - (u^*Zu + \alpha_m^* X^{-1}u + u^*(X^{-1})^*\alpha_m - \alpha_m^* X^{-1}Y\alpha_m)(a_m)\\
&= (\alpha_m^* X^{-1}Y\alpha_m)(b)\\
&\quad - (u^*Zu - u^*PX^\Delta X^{-1}u - u^*(X^{-1})^*(X^\Delta)^*Pu - u^*PX^\Delta X^{-1}Y(X^\Delta)^*Pu)(a_m)\\
&= (\alpha_m^* X^{-1}Y\alpha_m)(b)\\
&\quad - (u^*Zu - u^*ZXX^{-1}u - u^*(X^{-1})^*X^*Zu - u^*ZXX^{-1}Y(X^\Delta)^*Pu)(a_m)\\
&= (\alpha_m^* X^{-1}Y\alpha_m)(b)\\
&\quad - (u^*Zu - u^*Zu - u^*Zu - u^*ZY(X^\Delta)^*Pu)(a_m)\\
&= (\alpha_m^* X^{-1}Y\alpha_m)(b) + (u^*Zu + u^*ZY(X^\Delta)^*Pu)(a_m)
\end{aligned}
$$

Now X and Y are normalized conjoined bases of (4.1), and therefore,

$$
\begin{aligned}
I &= X^*[PY^\Delta] - [PX^\Delta]^*Y\\
&= X^*[PY^\Delta] - [ZX]^*Y\\
&= X^*[PY^\Delta] - X^*ZY.
\end{aligned}
$$

Rearranging, and multiplying on the left by $(X^*)^{-1}$, we get

$$ZY = PY^\Delta - (X^*)^{-1}.$$

We now substitute this into the expression from above to get

$$
\begin{aligned}
&\int_{a_m}^{b}\left[(u^\nabla)^*P^\rho u^\nabla - u^*Qu\right](t)\ \nabla t\\
&\quad\geq (\alpha_m^* X^{-1}Y\alpha_m)(b) + (u^*Zu + u^*[PY^\Delta - (X^*)^{-1}](X^\Delta)^*Pu)(a_m)\\
&\quad= (\alpha_m^* X^{-1}Y\alpha_m)(b) + (u^*Zu + u^*PY^\Delta(X^\Delta)^*Pu - u^*(X^*)^{-1}[PX^\Delta]^*u)(a_m)\\
&\quad= (\alpha_m^* X^{-1}Y\alpha_m)(b) + (u^*Zu + u^*PY^\Delta(X^\Delta)^*Pu - u^*(X^*)^{-1}[ZX]^*u)(a_m)\\
&\quad= (\alpha_m^* X^{-1}Y\alpha_m)(b) + (u^*Zu + u^*PY^\Delta(X^\Delta)^*Pu - u^*Zu)(a_m)\\
&\quad= (\alpha_m^* X^{-1}Y\alpha_m)(b) + (u^*PY^\Delta(X^\Delta)^*Pu)(a_m)\\
&\quad= (\alpha_m^* X^{-1}Y\alpha_m)(b) - (u^*PY^\Delta\alpha_m)(a_m)
\end{aligned}
$$

Then

$$\mathcal{F}[u] = \lim_{m\to\infty}\int_{a_m}^{b}[(u^\nabla)^*P^\rho u^\nabla - u^*Qu](t)\ \nabla t$$

$$\geq \lim_{m\to\infty} \left[(\alpha_m^* X^{-1} Y \alpha_m)(b) - (u^* P Y^{\Delta} \alpha_m)(a_m)\right]$$
$$= 0.$$

Finally, suppose $\mathcal{F}[u] = 0$. We wish to establish that this forces $w := P^\rho u^\nabla - Z^\rho u^\rho = 0$ on $(a, b]$. If we do so, then by the same reasoning used in the case where a was right-scattered, we will have $u(t) \equiv 0$, and we'll be done.

So, let

$$w_m = P^\rho u^\nabla - Z^\rho u^\rho - ((X^\rho)^{-1})^* \alpha_m.$$

Then we have

$$\lim_{m\to\infty} w_m = w$$

uniformly on $[a_k, b]$ for each $k \in \mathbb{N}$. Therefore,

$$\lim_{m\to\infty} \int_{a_k}^b w_m^* D w_m \ \nabla t = \int_{a_k}^b \lim_{m\to\infty} w_m^* D w_m \ \nabla t = \int_{a_k}^b w^* D w \ \nabla t.$$

Next, note that if $m \geq k$, then

$$\int_{a_m}^b w_m^* D w_m \ \nabla t \geq \int_{a_k}^b w_m^* D w_m \ \nabla t.$$

Letting $m \to \infty$, we then have

$$\lim_{m\to\infty} \int_{a_m}^b w_m^* D w_m \ \nabla t \geq \int_{a_k}^b w^* D w \ \nabla t.$$

But $\mathcal{F}[u] = 0$, and therefore

$$\lim_{m\to\infty} \int_{a_m}^b w_m^* D w_m \ \nabla t = 0,$$

which gives

$$0 = \lim_{m\to\infty} \int_{a_m}^b w_m^* D w_m \ \nabla t \geq \int_{a_k}^b w^* D w \ \nabla t \geq 0.$$

Finally, letting $k \to \infty$ gives

$$\int_a^b w^* D w \ \nabla t = 0.$$

Then we must have $w = 0$ on $(a, b]$, which completes the proof. □

Definition 117. We say that the self-adjoint matrix equation (4.1) is *disconjugate* on $[a, b]$ if the principal solution of (4.1) at a, $\tilde{X}$, has no focal points in $(a, b]$.

And now, finally, we are ready to prove Jacobi's condition.

Theorem 118 (Jacobi's Condition). *The self-adjoint matrix equation* (4.1) *is disconjugate if and only if the quadratic functional* $\mathcal{F}$ *is positive definite.*

Proof. First suppose (4.1) is disconjugate. Then the principal solution $\tilde{X}$, and the associated solution $\tilde{Y}$ satisfy the conditions of Theorem 116, and therefore $\mathcal{F} > 0$.

Now, suppose (4.1) is not disconjugate. Then there is $t_0 \in \mathbb{T}$ such that either

(i) $t_0 \in (a, b]$, and $\tilde{X}$ is invertible on (a, t_0), but $\tilde{X}(t_0)$ is singular, or

(ii) $t_0 \in (a, b]$, and $\tilde{X}$ is invertible on $(a, b]$, but

$$\tilde{D}(t_0) = \left[\tilde{X}^\rho \tilde{X}^{-1}(P^\rho)^{-1}\right](t_0) \text{ is not positive definite.}$$

Note that exactly one of these cases occurs. Now define the vector $d \in \mathbb{R}^n \setminus \{0\}$ as follows. If (i) occurs, choose d such that

$$\tilde{X}(t_0)d = 0.$$

If (ii) occurs, then there is a nonzero vector v such that

$$v^* \tilde{D}(t_0) v \leq 0.$$

Put

$$d = \tilde{X}^{-1}(t_0)(P^\rho)^{-1}(t_0)v.$$

Then we have, at t_0

$$\begin{aligned}
d^*(\tilde{X}^\rho)^* P^\rho \tilde{X} d &= v^*((P^\rho)^{-1})^*(\tilde{X}^{-1})^* \tilde{X}^{\rho *} P^\rho \tilde{X}\tilde{X}^{-1}(P^\rho)^{-1} v \\
&= v^*((P^\rho)^{-1})^*(\tilde{X}^{-1})^*(\tilde{X}^\rho)^* v \\
&= v^*(\tilde{X}^\rho \tilde{X}^{-1}(P^\rho)^{-1})^* v \\
&= v^* D^* v \\
&= v^* D v \\
&\leq 0.
\end{aligned}$$

Next, put

$$u(t) = \begin{cases} \tilde{X}(t)d & \text{if } t < t_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then by the way u is defined, we have

$$u^\nabla(t) = 0$$

for $t > t_0$, and therefore for $t > t_0$,

$$(u^\nabla)^* P^\rho u^\nabla - u^* Q u = 0.$$

Now, as $\tilde{X}$ is the principal solution of (4.1), we have $\tilde{X}(a) = 0$, and therefore $u(a) = u(b) = 0$. Note further that if either t_0 is left-scattered or t_0 is left-dense and (i) occurs, then $u \in \mathrm{C}^1_{\mathrm{pld}}$. We will deal with these possibilities first. If however, t_0 is left-dense and (ii) occurs, u may not be continuous at t_0, and we leave this scenario for the end of the proof.

So, assume first that t_0 is left-dense and (i) occurs. Then

$$\begin{aligned}
\mathcal{F}[u] &= \int_a^{t_0} \left((u^\nabla)^* P^\rho u^\nabla - u^* Q u\right)(t)\ \nabla t \\
&= \int_a^{t_0} \left(d^*(\tilde{X}^\nabla)^* P^\rho \tilde{X}^\nabla d - d^* \tilde{X}^* Q \tilde{X} d\right)(t)\ \nabla t \\
&= \int_a^{t_0} \left(d^*(\tilde{X}^\nabla)^* P^\rho \tilde{X}^\nabla d + d^* \tilde{X}^* [P\tilde{X}^\Delta]^\nabla d\right)(t)\ \nabla t \\
&= \int_a^{t_0} \left(d^*[(\tilde{X}^*)^\nabla (P\tilde{X}^\Delta)^\rho + \tilde{X}^*(P\tilde{X}^\Delta)^\nabla] d\right)(t)\ \nabla t \\
&= \int_a^{t_0} \left(d^*[\tilde{X}^* P \tilde{X}^\Delta]^\nabla d\right)(t)\ \nabla t \\
&= (d^* \tilde{X}^* P \tilde{X}^\Delta d)(t_0) - (d^* \tilde{X}^* P \tilde{X}^\Delta d)(a) \\
&= 0,
\end{aligned}$$

since $\tilde{X}(t_0)d = 0$ and $\tilde{X}(a) = 0$. Next suppose t_0 is left-scattered. Then we have

$$\begin{aligned}
\mathcal{F}[u] &= \int_a^{t_0} \left[(u^\nabla)^* P^\rho u^\nabla - u^* Q u\right](t)\ \nabla t \\
&= \int_a^{\rho(t_0)} \left[(u^\nabla)^* P^\rho u^\nabla - u^* Q u\right](t)\ \nabla t + \int_{\rho(t_0)}^{t_0} \left[(u^\nabla)^* P^\rho u^\nabla - u^* Q u\right](t)\ \nabla t
\end{aligned}$$

The first integral simplifies in similar fashion to the case above and we get

$$\begin{aligned}
\mathcal{F}[u] &= (d^* \tilde{X}^* P \tilde{X}^\Delta d)(\rho(t_0)) - (d^* \tilde{X}^* P \tilde{X}^\Delta d)(a) + [(u^\nabla)^* P^\rho u^\nabla - u^* Q u](t_0)\nu(t_0) \\
&= (d^*(\tilde{X}^\rho)^* P^\rho \tilde{X}^\nabla d)(t_0) + [(\nu u^\nabla)^* P^\rho (\nu u^\nabla)\nu^{-1}](t_0) \\
&= (d^*(\tilde{X}^\rho)^* P^\rho \tilde{X}^\nabla d)(t_0) + [(u - u^\rho)^* P^\rho (u - u^\rho)\nu^{-1}](t_0) \\
&= (d^*(\tilde{X}^\rho)^* P^\rho \tilde{X}^\nabla d)(t_0) + [(u^\rho)^* P^\rho u^\rho \nu^{-1}](t_0) \\
&= (d^*(\tilde{X}^\rho)^* P^\rho \tilde{X}^\nabla d + d^*(\tilde{X}^\rho)^* P^\rho \tilde{X}^\rho d \nu^{-1})(t_0) \\
&= (d^*(\tilde{X}^\rho)^* P^\rho [\nu \tilde{X}^\nabla - \tilde{X}^\rho] d \nu^{-1})(t_0) \\
&= (d^*(\tilde{X}^\rho)^* P^\rho \tilde{X} d \nu^{-1})(t_0) \le 0.
\end{aligned}$$

So, in either of these cases, we have $\mathcal{F}$ is not positive definite. We now consider the remaining case, (ii) occurs and t_0 is left-dense. Recall that we showed above that

when (ii) occurs, there is a nonzero vector d such that

$$(d^*(\tilde{X}^\rho)^* P^\rho \tilde{X} d)(t_0) \leq 0.$$

Then, since t_0 is left-dense, we have

$$(d^* \tilde{X}^* P \tilde{X} d)(t_0) \leq 0.$$

Now put

$$c = \tilde{X}(t_0)d.$$

Since $\tilde{X}(t_0)$ is invertible, $c \neq 0$, and furthermore, $c^*P(t_0)c \leq 0$. In fact, since $P(t_0)$ is invertible, we have $c^*P(t_0)c < 0$. We seek to show that $\mathcal{F}$ is not positive definite, so, by way of contradiction, assume it is. First assume t_0 is right-scattered, and let $\{t_m\}_{m=1}^{\infty} \subset (a, t_0)$ be an increasing sequence of points with $\lim_{m\to\infty} t_m = t_0$. Now, for $m \in \mathbb{N}$, put

$$u_m(t) = \begin{cases} \frac{t-t_m}{\sqrt{t_0-t_m}}c & \text{if } t \in [t_m, t_0] \\ 0 & \text{otherwise.} \end{cases}$$

Then since t_0 is right-scattered, we have $t_0 < b$, so $u_m(a) = u_m(b) = 0$. Also, $u_m \in \mathrm{C}^1_{\mathrm{pld}}$, and $u(t_0) = \sqrt{t_0 - t_m}c \neq 0$, so u is nontrivial. Then we see that

$$\begin{aligned}
0 < \mathcal{F}[u_m] &= \int_{t_m}^{\sigma(t_0)} \left\{(u_m^\nabla)^* P^\rho u_m^\nabla - u_m^* Q u_m\right\}(t)\, \nabla t \\
&= \int_{t_m}^{t_0} \left\{(u_m^\nabla)^* P^\rho u_m^\nabla - u_m^* Q u_m\right\}(t)\, \nabla t \\
&\quad + \int_{t_0}^{\sigma(t_0)} \left\{(u_m^\nabla)^* P^\rho u_m^\nabla - u_m^* Q u_m\right\}(t)\, \nabla t \\
&= \int_{t_m}^{t_0} c^* \left(\frac{1}{\sqrt{t_0 - t_m}}\right) P^\rho(t) \left(\frac{1}{\sqrt{t_0 - t_m}}\right) c\, \nabla t \\
&\quad - \int_{t_m}^{t_0} c^* \left(\frac{t - t_m}{\sqrt{t_0 - t_m}}\right) Q(t) \left(\frac{t - t_m}{\sqrt{t_0 - t_m}}\right) c\, \nabla t \\
&\quad + \left\{(u_m^\nabla)^* P^\rho u_m^\nabla - u_m^* Q u_m\right\}(\sigma(t_0))\nu(\sigma(t_0)) \\
&= \int_{t_m}^{t_0} c^* \left(\frac{1}{t_0 - t_m}\right) P^\rho(t) c\, \nabla t - \int_{t_m}^{t_0} c^* \left(\frac{(t - t_m)^2}{t_0 - t_m}\right) Q(t) c\, \nabla t \\
&\quad + \left[u_m^\nabla(\sigma(t))\right]^* P(t_0) \left[u_m^\nabla(\sigma(t_0))\right] \nu(\sigma(t_0))
\end{aligned}$$

We now want to look at what happens to each of these terms as $m \to \infty$. Looking at the third term, we see that

$$u_m^\nabla(\sigma(t_0)) = \frac{u_m(\sigma(t_0)) - u_m(t_0)}{\mu(t_0)} = \frac{0 - \left(\frac{t_0 - t_m}{\sqrt{t_0 - t_m}}\right) c}{\mu(t_0)} = -\frac{\sqrt{t_0 - t_m}\, c}{\mu(t_0)}.$$

So, the third term, then, is

$$c^*\frac{t_0 - t_m}{\mu(t_0)}P(t_0)c,$$

which goes to 0 as $m \to \infty$. Next, we consider the second term, the integral containing Q. It is easy to see that this term also goes to 0 as $m \to \infty$. Finally, we turn our attention to the first term, the integral containing P^ρ. Using L'Hôpital's rule we get

$$\begin{aligned}\lim_{m\to\infty}\int_{t_m}^{t_0} c^*\left(\frac{1}{t_0 - t_m}\right)P^\rho(t)c\ \nabla t &= \lim_{m\to\infty}\frac{\int_{t_m}^{t_0} c^*P^\rho(t)c^*\ \nabla t}{t_0 - t_m}\\ &= \lim_{m\to\infty}\frac{-c^*P^\rho(t_m)c}{-1}\\ &= c^*P(t_0)c.\end{aligned}$$

Thus, we have

$$0 \leq \lim_{m\to\infty}\mathcal{F}[u_m] = c^*P(t_0)c < 0,$$

which is a contradiction. The remaining case, t_0 right-dense, is handled similarly. Let $\{s_m\}_{m=1}^\infty$ be a decreasing sequence with $\lim_{m\to\infty} s_m = t_0$, then choose an increasing sequence, $\{t_m\}_{m=1}^\infty$ such that $\lim_{m\to\infty} t_m = t_0$ and such that for each $m \in \mathbb{N}$, $t_0 - t_m \leq s_m - t_0$. Applying the same technique as above using

$$u_m(t) = \begin{cases} \frac{t-t_m}{\sqrt{t_0-t_m}}c & \text{if } t \in [t_m, t_0]\\ \frac{(s_m-t)\sqrt{t_0-t_m}}{s_m-t_0}c & \text{if } t \in (t_0, s_m]\\ 0 & \text{otherwise}\end{cases}$$

results in a similar contradiction. □

Definition 119. We call a solution, X, of (4.1) a *basis* whenever

$$\operatorname{rank}\begin{bmatrix} X(a)\\ P(a)X^\Delta(a)\end{bmatrix} = n.$$

We now get the following version of Sturm's separation theorem.

Theorem 120 (Sturm Separation Theorem). *If there is a conjoined basis of* (4.1) *with no focal points in* $(a, b]$*, then equation* (4.1) *is disconjugate on* $[a, b]$.

Proof. Let X be a conjoined basis of (4.1) with no focal points in $(a, b]$. Since X is a basis, we have that the matrix

$$A = X^*(a)X(a) + (X^\Delta)^*(a)P^2(a)X^\Delta(a)$$

is invertible. Now, let Y be the solution of the IVP

$$LY = 0, \quad Y(a) = -P(a)X^\Delta(a)A^{-1}, \quad Y^\Delta(a) = P^{-1}(a)X(a)A^{-1}.$$

Then by Abel's formula, we have

$$\begin{aligned}
W(Y,Y)(t) &= W(Y,Y)(a) \\
&= \left[Y^*PY^\Delta - (Y^\Delta)^*PY\right](a) \\
&= -(A^{-1})^*(X^\Delta)^*(a)P^2(a)P^{-1}(a)X(a)A^{-1} \\
&\quad + (A^{-1})^*X^*(a)P^{-1}(a)P^2(a)X^\Delta(a)A^{-1} \\
&= (A^{-1})^*\left[(X^\Delta)^*PX - X^*PX^\Delta\right](a)A^{-1} \\
&= 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
W(X,Y)(t) &= W(X,Y)(a) \\
&= \left[X^*PY^\Delta - (X^\Delta)^*PY\right](a) \\
&= X^*(a)P(a)P^{-1}(a)X(a)A^{-1} + (X^\Delta)^*(a)P^2(a)X^\Delta(a)A^{-1} \\
&= \left[X^*X + (X^\Delta)^*P^2X^\Delta\right](a)A^{-1} \\
&= AA^{-1} \\
&= I.
\end{aligned}$$

Therefore, we see that X and Y are normalized conjoined bases of (4.1). Then by Theorem 116, $\mathcal{F}$ is positive definite. Applying Jacobi's condition then gives us that (4.1) is disconjugate, as desired. □

We conclude by establishing an analogue of Sturm's Comparison Theorem. Consider the equation

$$\left[\tilde{P}X^\Delta\right]^\nabla + \tilde{Q}X = 0, \tag{4.5}$$

where $\tilde{P}$ and $\tilde{Q}$ satisfy the same assumptions as P and Q, respectively.

Theorem 121. *Suppose that for all $t \in \mathbb{T}$, we have*

$$\tilde{P}(t) \leq P(t) \quad \text{and} \quad \tilde{Q}(t) \geq Q(t).$$

Then if (4.5) is disconjugate then (4.1) is also disconjugate.

Proof. Suppose (4.5) is disconjugate, let $u \in \mathrm{C}^1_{\mathrm{pld}}$ with $u(a) = u(b) = 0$, and assume u is nontrivial. Then by Jacobi's condition,

$$\tilde{\mathcal{F}}[u] := \int_a^b \left[(u^\nabla)^*\tilde{P}^\rho u^\nabla - u^*\tilde{Q}u\right](t)\ \nabla t > 0.$$

Therefore

$$\begin{aligned}\mathcal{F}[u] &= \int_a^b \left[(u^\nabla)^* P^\rho u^\nabla - u^* Q u\right](t)\ \nabla t \\ &\geq \int_a^b \left[(u^\nabla)^* \tilde{P}^\rho u^\nabla - u^* \tilde{Q} u\right](t)\ \nabla t \\ &= \tilde{\mathcal{F}}[u] > 0.\end{aligned}$$

Therefore $\mathcal{F}$ is positive definite, and by Jacobi's condition, (4.1) is disconjugate.

□

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